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Functional analysis

A Electro-Viscoelastic Contact Problem Variational analysis of an electro-viscoelastic problem with an electrically conductive foundation: Duale Forms

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1. Introduction

La mécanique du contact est un sujet trés vaste, qui embrasse plusieurs phénomènes de contact impliquant des corps déformables abondent en industrie et dans la vie de tous les jours. Le simple contact entre le piston avec la chemise, la roue avec le rail et d'une chaussure avec le sol ne représentent que trois exemples parmi bien d'autres. Compte tenu du fait que ces phénomènes jouent un rôle important dans les structures et les systèmes mécaniques, ils ont été intensivement étudiés depuis longue date et la littérature relevant des Sciences de l'Ingénieur qui leur est dédiée est assez riche.

La piézoélectricité (du grec piézein presser, appuyer) est la propriété que possèdent certains corps de se polariser électriquement sous l'action d'une contrainte mécanique et réciproquement de se déformer lorsqu'on leur applique un champ électrique. Les deux effets sont indissociables. Le premier est appelé effet piézoélectrique direct, le second effet piézoélectrique inverse. Cette propriété trouve un très grand nombre d'applications dans l'industrie et la vie quotidienne. Une application parmi les plus familières est l'allume-gaz. Dans un allume-gaz, la pression exercée produit une tension électrique qui se décharge brutalement sous forme d'étincelles : c'est une application de l'effet direct. De manière plus générale, l'effet direct peut être mis à profit dans la réalisation de capteurs (capteur de pression etc.) tandis que l'effet inverse permet de réaliser des actionneurs (injecteurs à commande piézoélectrique en automobile, nanomanipulateur...).

Le but de ce mémoire l'étude mathématique d'un problème de contact et une base rigide, en tenant compte de l'effet piézoélectrique voir par exemple [6]. Sous l'hypothése des petites déformations, nous étudions le processus quasistatique pour de matériaux électro-viscoélastiques [9], ceci constitue une généralisation de l'article [4] et [11]. Les conditions aux limites sont de type Signorini. Les conditions électriques sont introduites dans le cas où la fondation est conductrice. Les résultats que nous présentons dans ce travail sont essentiellement des résultats d'existence et d'unicité de la solution (forme dual) . De plus pour l'étude de ce problème, nous utilisons essentiellement des méthodes standard sur les inéquations variationnelles elliptiques, des résultats de monotonie, de convexité et de point fixe.

Problème $\mathcal{P}$: *Trouver le champ des déplacements* $u : \Omega \times (0,T) \longrightarrow \mathbb{R}^d$, *le champ des contraintes* $\sigma : \Omega \times (0,T) \longrightarrow \mathbb{S}^d$, *un potentiel électrique* $\varphi : \Omega \times (0,T) \to \mathbb{R}$, *un champ de déplacement électrique* $\mathbf{D} : \Omega \times (0,T) \to \mathbb{R}^d$ *tels que:*

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{G}\varepsilon(u) - \mathcal{E}^* E(\varphi) \quad \text{dans } \Omega \times (0,T) \tag{1}$$

$$\mathbf{D} = \mathcal{E}\varepsilon(u) + \mathcal{B}E(\varphi) \quad \text{dans } \Omega \times (0,T) \tag{2}$$

$$Div\sigma + f_0 = 0 \quad \text{dans } \Omega \times (0,T) \tag{3}$$

$$\div D = q_0 \quad \text{dans } \Omega \times (0,T) \tag{4}$$

$$u = 0 \quad \text{sur } \Gamma_1 \times (0,T) \tag{5}$$

$$\sigma\nu = f_2 \quad \text{sur } \Gamma_2 \times (0,T) \tag{6}$$

$$\begin{cases} u_\nu \leq 0 \,,\ \sigma_v \leq 0 \,, \\ \sigma_\tau = 0 \end{cases} \quad \text{sur } \Gamma_3 \times (0,T) \tag{7}$$

$$\varphi = 0 \text{ sur } \Gamma_a \times (0,T) \tag{8}$$

$$D.\nu = q_2 \text{ sur } \Gamma_b \times (0,T) \tag{9}$$

$$D.\nu = \psi(u_\nu)\,\phi(\varphi - \varphi_0) \text{ sur } \Gamma_3 \times (0,T) \tag{10}$$

$$u(0) = u_0 \text{ sur } \Omega \times (0,T) \tag{11}$$

2. Formulation duale

Nous donnons dans cette section une formulation duale du problème $\mathcal{P}$, exprimée en terme de contrainte et de champ des déplacements électriques. Cette formulation faible établie, nous présentons deux résultats dont le premier concerne l'existence et l'unicité d'une solution faible. Pour définir la formulation duale du problème$\mathcal{P}$, on considére l'espace de Hilbert $Y = \mathcal{H} \times L^2(\Omega)^d$,muni de produit scalaire donné par

$$(x,y) = (\sigma,\tau)_{\mathcal{H}} + (D,E)_{L^2(\Omega)^d} \ \ \forall x = (\sigma, D), y = (\tau, E) \in Y$$

soit $\Sigma_{ad}(t)$ l'ensemble des "contraintes admissibles" donné par

$$\Sigma_{ad}(t) = \{\tau \in \mathcal{H} \ / \ (\tau, \varepsilon(v))_{\mathcal{H}} \geq (f(t), v)_V \quad \forall v \in U_{ad}\}. \tag{12}$$

Nous utilisons également la notation $\Im$ pour désigner l'ensemble

$$\Im(t) = \left\{ E \in L^2(\Omega)^d \ / \ (E, \nabla\psi)_{L^2(\Omega)^d} + (h(u,\varphi), \psi)_{L^2(\Omega)^d} = (q(t), \psi)_W \ \ \forall \psi \in W \right\}.$$

Commençons tout d'abord par un lemme, qui va nous permettre d'introduire la formulation duale du problème $\mathcal{P}$. **lemma** Si (u, σ, D, φ) est une solution classique du problème $\mathcal{P}$, alors (σ, D) satisfait

$$\sigma \in \Sigma_{ad}(t), \ (\tau - \sigma, \varepsilon(\dot{u}(t)))_{\mathcal{H}} \geq 0, \ \ \forall \tau \in \Sigma_{ad}(\sigma) \tag{13}$$

$$-D \in \Im(t), \ \ (E + D, \nabla\varphi)_{L^2(\Omega)^d} = 0, \ \forall E \in \Im. \tag{14}$$

Probléme $\mathcal{PV}^d$.Trouver le champ des contraintes $\sigma : [0,T] \to \mathcal{H}$ et le champ des déplacements électriques $D : [0,T] \to L^2(\Omega)^d$ tels que

$$\sigma \in \Sigma_{ad}(t) \ \ (K(\sigma, -D, \varepsilon(u_0)), \tau - \sigma)_{\mathcal{H}} \geq 0 \ \forall \ \tau \in \Sigma_{ad}(t). \tag{15}$$

$$-D \in \Im \ \ (H(\sigma, -D, \varepsilon(u_0)), E + D)_{L^2(\Omega)^d} = 0 \ \forall \ E \in \Im. \tag{16}$$

3. theorème

On considère (1)-(16), le problème $\mathcal{PV}^d$ possède une solution unique $(\sigma, -D) \in C([0;T]; \mathcal{H}_1 \times \mathcal{W}_1)$.

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A FIFTH-ORDER KADOMTSEV-PETVIASHVILI II EQUATION IN ANISOTROPIC GEVREY SPACES

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We show that the fifth-order Kadomtsev-Petviashvili II equation

$$\begin{cases} \partial_t u - \partial_x^5 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \\ u(x,y,0) = f(x,y), \end{cases} \tag{1}$$

where $u = u(x,y,t)$ and $(x,y,t) \in \mathbb{R}^3$.

is globally well-posed in an anisotropic Gevrey space $G^{\sigma_1,\sigma_2}(\mathbb{R}^2)$, which complements earlier results on the well-posedness of this equation in anisotropic Sobolev spaces [1].

With

$$\|f\|_{G^{\sigma_1,\sigma_2}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} e^{2\sigma_1|\xi|}e^{2\sigma_2|\eta|}|\tilde{f}(\xi,\eta)|^2 d\xi d\eta\right)^{1/2}.$$

The method used here for proving lower bounds on the radius of analyticity. The main function spaces they used are the so-called Bourgain spaces, whose norm is given by

$$\|u\|_{X^{s_1,s_2,b,\varepsilon}_{\sigma_1,\sigma_2}} = \left(\int_{\mathbb{R}^3} e^{2\sigma_1|\xi|}e^{2\sigma_2|\eta|}\lambda^2(s_1,s_2,b,\varepsilon)|\hat{u}(\xi,\eta,\tau)|^2 d\xi d\eta d\tau\right)^{\frac{1}{2}},$$

where

$$\lambda(s_1,s_2,b,\varepsilon) = \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\langle\tau - m(\xi,\eta)\rangle^b \left\langle\frac{\tau - m(\xi,\eta)}{1+|\xi|^5}\right\rangle^{\varepsilon}.$$

with $m(\xi,\eta) = \xi^5 - \frac{\eta^2}{\xi}$.

Keywords:KPII equation, Gevrey space, radius of spatial analyticity
2010 Mathematics Subject Classification: 35Q35, 35Q53

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A FIXED POINT THEOREM IN b-METRIC SPACES

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Abstract. In this presentation, we prove a fixed point theorem for contractive mapping has unique fixed point in the context of b-metric spaces. Also, we present an example to illustrate the validity of the result obtained in the presentation.

2010 Mathematics Subject Classification. xxxx, xxxx, xxxx.

Keywords and phrases: fixed point, contractive mappings , b-metric space .

1. Fixed point theorem

In this section, we several fixed point theorem for contractive mappings on complete b-metric spaces.

Theorem 1.1. *Let (X, d) be a complete b-metric space with a constant $s \geq 1$ and $f : X \to X$ be a mapping on X. Suppose that q are nonnegative reals with $q < 1$, such that the inequality*

$$sd(fx, fy) \leq q \max \left\{ d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)} \right\}, \tag{1}$$

holds for each $x, y \in X$. Then f has a unique fixed point.

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A GENERAL DECAY AND OPTIMAL DECAY RESULT IN A HEAT SYSTEM WITH A VISCOELASTIC TERM

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Abstract : We consider a quasilinear heat system in the presence of an integral term and establish a general and optimal decay result from which improves and generalizes several stability results in the literature.

Key words : heat equation; viscoelastic; general decay; optimal

Classification MSC2010 : 35K05

1 Introduction

In this work, we consider the following problem

$$\begin{cases} A(t)\,|u_t|^{m-2}\,u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s = 0, & \Omega\times(0,+\infty), \\ u(x,t)=0, & \partial\Omega\times\mathbb{R}^+, \\ u(x,0)=u_0(x), & \Omega, \end{cases} \tag{1}$$

where $m\geq 2,\Omega$ is a bounded domain of $\mathbb{R}^n, n\in\mathbb{N}^*:=\{1,2,\cdots\}$, with a smooth boundary $\partial\Omega, g:\mathbb{R}^+\to\mathbb{R}^+$ is a positive nonincreasing function, and

$$A:\mathbb{R}^+\to M_n(\mathbb{R})$$

is a bounded square matrix satisfying $A\in C\left(\mathbb{R}^+\right)$ and, for some positive constant c_0,

$$(A(t)v,v)\geq c_0|v|^2,\quad \forall t\in\mathbb{R}^+,\forall v\in\mathbb{R}^n,$$

where $(\cdot,\cdot)$ and $|\cdot|$ are the inner product and the norm, respectively, in $\mathbb{R}^n$. The equation in consideration arises from various mathematical models in engineering and physics.
In this work, we discuss (1) when g is of a more general decay, and establish a general and optimal decay result, which improves those of Berrimi and Messaoudi [1], Liu and Chen [2], and Messaoudi and Tellab [3]. For the relaxation function g we assume that
(G1) The function $g:\mathbb{R}^+\to\mathbb{R}^+$ is a differentiable function satisfying

$$g(0)>0\quad\text{and}\quad 1-\int_0^{+\infty}g(s)\mathrm{d}s=l>0.$$

(G2) There exist a constant $p\in[1,3/2)$ and a nonincreasing differentiable function $\xi:\mathbb{R}^+\to\mathbb{R}^+$ satisfying

$$g'(t)\leq-\xi(t)g^p(t),\quad\forall t\in\mathbb{R}^+.$$

(G3) We also assume that

$$2\leq m\leq\frac{2n}{n-2},\quad\text{if}\quad n\geq 3,\qquad m\geq 2\qquad\text{if}\quad n=1,2.$$

Our main result is the following

Theorem 1 *Let u be the solution of* (1). *Then, there exist two strictly positive constants λ_0 and λ_1 such that the energy satisfies, for all $t \in \mathbb{R}^+$,*

$$E(t) \leq \lambda_0 \mathrm{e}^{-\lambda_1 \int_0^t \xi(s)\mathrm{d}s}, \quad \text{if} \quad p = 1,$$
$$E(t) \leq \lambda_0 \left(1 + \int_0^t \xi^{2p-1}(s)\mathrm{d}s\right)^{\frac{-1}{2p-2}}, \quad \text{if} \quad p > 1.$$

Moreover, if ξ and p in (G2) satisfy

$$\int_0^{+\infty} \left(1 + \int_0^t \xi^{2p-1}(s)\mathrm{d}s\right)^{\frac{-1}{2p-2}} \mathrm{d}t < +\infty,$$

then, for all $t \in \mathbb{R}^+$.

$$E(t) \leq \lambda_0 \left(1 + \int_0^t \xi^{p}(s)\mathrm{d}s\right)^{\frac{-1}{p-1}}, \quad \text{if} \quad p > 1.$$

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A DYNAMIC FRICTIONAL CONTACT PROBLEMS GOVERNED BY A VARIATIONAL AND HEMIVARIATIONAL INEQUALITIES IN VISCOELASTICITY

L. AIT KAKI

ABSTRACT. We consider a dynamic problem that describes a frictional contact with damage between a viscoelastic body and a foundation. The contact is supposed bilateral and frictional, which includes the damage effects and consider a nonmonotone and multivalued subdifferential boundary conditions for the contact friction flux. The model consists of the system of the hemivariational inequality of hyperbolic type for the displacement and the parabolic variational inequality for the damage. The existence of solutions is proved by using some results from the theory of hemivariational inequalities, evolutionary variational inequalities, and fixed point arguments.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 47J20, 47J22, 49J40, 49J45.

KEYWORDS AND PHRASES. Evolutionary variational inequality, Fixed point, Frictional contact, Hemivariational inequality.

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A FINITE-TIME BLOW-UP RESULT FOR A CLASS OF SOLUTIONS WITH POSITIVE INITIAL ENERGY FOR COUPLED SYSTEM OF HEAT EQUATIONS WITH MEMORIES

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ABSTRACT. In this work, we are interested by a system of heat equations with initial condition and zero Dirichlet boundary conditions. We prove a finite-time blow-up result for a large class of solutions with positive initial energy.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 35K05; 35B44; 93D20; 93C10.

KEYWORDS AND PHRASES. blow up, heat equation, positive initial energy.

1. DEFINE THE PROBLEM

Here, we are going to study the blow up in finite time of solutions for the following system:

$$
\begin{cases}
u' - \Delta_x u +_0^t \eta_1(t-s)\Delta_x u(s)ds = f(u,v) & \text{in } \Omega \times (0,T),\\
v' - \Delta_x v +_0^t \eta_2(t-s)\Delta_x v(s)ds = g(u,v) & \text{in } \Omega \times (0,T),\\
(u,v) = (0,0) & \text{on } \partial\Omega \times (0,T),\\
(u(0), v(0)) = (u_0, v_0) & \text{in } \Omega,
\end{cases}
\tag{1}
$$

where Ω is bounded domain in $\mathbb{R}^n$, $n \geq 1$ with smooth boundary $\partial\Omega$ and r is a real constant satisfies

$$
\begin{cases}
r > 2 & \text{if } n = 1; 2,\\
2 < r < \frac{2(n-1)}{n-2} & \text{if } n \geq 3.
\end{cases}
\tag{2}
$$

and

$$
\begin{aligned}
f(u,v) &= |u+v|^{r-2}(u+v) + |u|^{\frac{r-4}{2}}u|v|^{\frac{r}{2}},\\
g(u,v) &= |u+v|^{r-2}(u+v) + |v|^{\frac{r-4}{2}}v|u|^{\frac{r}{2}}.
\end{aligned}
$$

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A GLOBAL SOLUTION TO A MASS CONSERVED ALLEN CAHN PROBLEM

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ABSTRACT. We attempt to prove the existence and uniqueness of a global solution for an Allen Chan mass conserved problem, which models a phase transition. The proof relies on the monotonicity method where the nonlinear diffusion operator satisfies the properties of monotonicity.

KEYWORDS AND PHRASES. Allen Cahn problem, Mass conservation, Monotonicity method.

1. Define the problem

We consider here a reaction-diffusion equation given by a nonlocal mass conserved Allen-Cahn problem, which models a phase transition in binary mixture.

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A NONLINEAR BOUNDARY VALUE PROBLEM INVOLVING A MIXED FRACTIONAL DIFFERENTIAL EQUATION

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ABSTRACT. In this work, we discuss a special type of nonlinear boundary value problems which involves both right-sided Caputo-Katugampola and left-sided Katugampola fractional derivatives. To study the existence and uniqueness of solutions for the aforementioned problem, we have first written it in the form of a Volterra integral equation then we have used Banach's contraction principle.

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KEYWORDS AND PHRASES. Volterra integral equation, mixed fractional derivatives, nonlinear boundary value problems.

1. DEFINE THE PROBLEM

Our objective in this work is to discuss the existence and uniqueness of solution for the following nonlinear BVP:

$$ {}^{C}D_{1^-}^{\beta,\rho}\left(D_{0^+}^{\alpha,\rho}y\right)(t) = g(t,y(t)), \quad t\in[0,1], \tag{1}$$

with

$$\left(I_{0^+}^{1-\alpha,\rho}y\right)(0) = 0, \tag{2}$$

$$\left(D_{0^+}^{\alpha,\rho}\right)y(1) = 0, \tag{3}$$

where $\alpha,\beta\in(0,1)$, $\rho>0$, ${}^{C}D_{1^-}^{\beta,\rho}$ is the right Caputo-Katugampola fractional derivative of order β. $D_{0^+}^{\alpha,\rho}$ is the left Katugampola fractional derivative of order β and $I_{0+}^{1-\alpha,\rho}$ is the Katugampola fractional integral.

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A PROBLEM OF MINIMIZATION SUBJECT TO A DIFFERENTIAL INCLUSION BY THE SUBDIFFERENTIAL

FENNOUR FATIMA AND SOUMIA SAÏDI

Abstract. The paper proposes a minimization problem subject to a differential inclusion governed by the subdifferential of a time-dependent proper convex lower semi-continuous function with a single-valued perturbation.

2010 Mathematics Subject Classification. 34A60, 49J52, 49J53

Keywords and phrases. Differential inclusion, subdifferential operator, optimal solution.

1. Statement of the problem

Let $I := [0, T]$. For each $t \in I$, let $\varphi(t, \cdot)$ be a proper lower semi-continuous and convex function. Set $L := \{h \in L^2_{\mathbb{R}}(I) : \ |h(t)| \leq 1 \text{ a.e.}\}$. Let $J : I \times \mathbb{R} \times \mathbb{R} \longrightarrow [0, +\infty[$ be measurable and such that $J(t, \cdot, \cdot)$ is lower semi-continuous on $\mathbb{R} \times \mathbb{R}$ for every $t \in I$, and $J(t, x, \cdot)$ is convex on $\mathbb{R}$ for every $(t, x) \in I \times H$.
Then, we prove that the minimization problem

$$\min_{h \in L} \int_0^T J\big(t, u_h(t), h(t)\big) dt$$

subject to

$$(P_h) \begin{cases} -\dot{u}_h(t) \in \partial\varphi(t, u_h(t)) + h(t), & \text{a.e.}\, t \in I \\ u_h(t) \in dom\varphi(t, \cdot), & \forall\, t \in I \\ u_h(0) = u_0 \in dom\varphi(0, \cdot) \end{cases}$$

has an optimal solution, where u_h denotes the unique absolutely continuous solution associated with the control $h \in L$.

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ABOUT PROPERTIES OF MEROMORPHIC SOLUTIONS OF ULTRAMETRIC q-DIFFERENCE EQUATIONS OF SHRÖDER-TYPE

SALIH BOUTERNIKH AND TAHAR ZERZAIHI

Abstract. Let $\mathbb{K}$ be an algebraically closed field, complete for an ultrametric absolute value, let $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of entire functions in $\mathbb{K}$ and $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$ i.e. the field of fonctions f such that $f = h/g$, with $h, g \in \mathcal{A}(d(0, R^-))$.

We investigate the growth of transcendental meromorphic solutions of some ultrametric q-difference equations and find the order of growth of these solutions. our method is based on the ultrametric Nevanlinna theory.

2010 Mathematics Subject Classification. 30G06, 11J97, 12H10.

Keywords and phrases. Ultrametric meromorphic functions, Value distribution theory, q-Difference equations.

1. Define the problem

In this paper, we consider the ultrametric functional equation of Shröder-type: $\sum_{j=1}^{n} A_j(x) f(q^j x) = R(x, f(x)) = \dfrac{P(x, f(x))}{Q(x, f(x))}$, where q is an element of $\mathbb{K}$, $A_1(x), \ldots, A_n(x)$ are rational functions and P, Q are relatively prime polynomials in f over the field of rational functions satisfying $p = \deg_f P$, $t = \deg_f Q$, $d = p - t \geqslant 2$.

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ANALYSIS OF A FRICTIONAL CONTACT PROBLEM WITH ADHESION FOR PIEZOELECTRIC MATERIALS

LATRECHE SOUMIA AND SELMANI LYNDA

Abstract. This work is devoted to the study of the mathematical model involving a quasistatic frictional contact between an electro-elasto-viscoplastic body and a conductive adhesive foundation. The contact is described with a normal compliance condition with adhesion, the associated general version of Coulombs law of dry friction in which the adhesion of contact surfaces is taken into account and a regularized electrical conductivity condition. We derive a variational formulation of the problem and state that, under a smallness assumption on the surface conductance, there exists a unique weak solution for the model. The proof is based on arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem.

2010 Mathematics Subject Classification. 74M15, 74M10, 74F15, 74D10.

Keywords and phrases. Electro-elasto-viscoplastic materials, quasistatic process, internal state variable, frictional contact, normal compliance, adhesion, weak solution, fixed point.

1. Introduction

The aim of this work consists on the study of a contact problem for piezoelectric materials. We investigate a mathematical model which describes the frictional contact between a deformable body assumed to be electroelasto-viscoplastic with internal state variable and a conductive adhesive foundation. The contact is modeled with a normal compliance condition, the associated general version of Coulombs law of dry friction in which the adhesion is taken into account and a regularized electrical conductivity condition.
We deal with the study of a quasistatic problem of frictional adhesive contact for general electro-elasto-viscoplastic materials of the form

$$\begin{aligned}\boldsymbol{\sigma}(t) =&\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \mathcal{E}^*\mathbf{E}(\varphi(t))\\ &+ \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)) + \mathcal{E}^*\mathbf{E}(\varphi(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{k}(s))ds, \quad (1)\end{aligned}$$

$$\dot{\boldsymbol{k}}(t) = \phi(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{E}^*\mathbf{E}(\varphi(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{k}(t)), \quad (2)$$

$$\mathbf{D}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \mathbf{B}\nabla\varphi(t), \quad (3)$$

where $\boldsymbol{u}$ is displacement field, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ are the stress and the linearized strain tensor, respectively. $\mathbf{D}$ is the electric displacement field. Here $\mathcal{A}$ and $\mathcal{F}$ are operators describing the purely viscous and the elastic properties of the material, respectively. $\mathcal{G}$ is a nonlinear constitutive function describing

the viscoplastic behavior of the material and depending on the internal state variable $\boldsymbol{k}$ and ϕ is also a nonlinear constitutive function which depends on $\boldsymbol{k}$. $\mathbf{E}$ is the electric field that satisfies $\mathbf{E}(\varphi) = -\nabla\varphi$, where φ is the electric potential. Also, $\mathcal{E}$ represents the third order piezoelectric tensor, $\mathcal{E}^*$ is its transposed and $\mathbf{B}$ denotes the electric permittivity tensor. Frictional and frictionless contact problems involving electro-elasto-viscoelastic constitutive law were studied in [1, 5].
We assume the decomposition of the form $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{EVP} + \boldsymbol{\sigma}^{E}$, where $\boldsymbol{\sigma}^{E} = -\mathcal{E}^*\mathbf{E}(\varphi) = \mathcal{E}^*\nabla\varphi$ is the electric part of the stress and $\boldsymbol{\sigma}^{EVP}$ is the elastic-viscoplastic part of the stress which satisfies

$$\dot{\boldsymbol{k}}(t) = \phi(\boldsymbol{\sigma}^{EVP}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{k}(t)).$$

A frictionless contact for elastic-viscoplastic materials with or without internal state variable were considered in [3, 4].
When $\mathcal{G} = 0$ the constitutive law (1)-(3) reduces to the electro-viscoelastic law given by (3) and

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{E}^*\nabla\varphi(t).$$

A frictional contact problem for an electro-viscoelastic body was considered in [2].
When $\mathcal{G} = 0$ and $\mathcal{A} = 0$ the constitutive law (1)-(3) becomes the electro-elastic constitutive law given by (3) and

$$\boldsymbol{\sigma}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{E}^*\nabla\varphi(t).$$

This work is structured as follows. First, we present notation and some preliminaries. Then, we give the mathematical model of the problem and the variational formulation. Finally, we present our main result and its proof which is based on arguments of time-dependent variational inequalities, differential equations and fixed point.

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APPLICATION OF FIXED POINT THEOREM FOR STUDY EXISTENCE OF POSITIVE SOLUTIONS FOR BOUNDARY VALUE PROBLEMS.

ZOUAOUI BEKRI AND SLIMANE BENAICHA

Abstract. In this paper, we applied the Leray-Schauder nonlinear alternative and Leray-Schauder fixed point theorem for study existence of positive solutions for fifth-order boundary value problem of the form

$$u^{(5)}(t) = q(t)f(t, u(t), u^{'}(t), u^{''}(t), u^{'''}(t), u^{(4)}(t)),\ 0 < t < 1,$$
$$u(0) = u^{'}(1) = u^{''}(0) = u^{'''}(1) = u^{(4)}(0) = 0,$$

where $f \in C([0,1]\times[0,\infty)\times[0,\infty)\times(-\infty,0]\times(-\infty,0]\times[0,\infty) \to [0,\infty))$. As an application, we also given an example to illustrate the results obtained.

2010 Mathematics Subject Classification. 34B15, 34B18.

Keywords and phrases. Green's function, Positive solution, Leary-Schauder nonlinear alternative, Fixed point theorem, Boundary value problem.

1. Introduction

Fixed point theorems are the basic mathematical tools in showing the existence of solutions in various kinds of equations. The fixed point theory is at the heart of nonlinear analysis as it provides the tools necessary to have existence theorems in many different nonlinear problems. She uses her tools of analysis and topology and for this reason we have the classification fixed point and metric theory and fixed point and topological theory.

Motivated by the above works, the aim of this paper is to apply the Leray-Schauder nonlinear alternative and Leray-Schauder fixed point theorem for study existence of positive solutions for fifth-order boundary value problem

$$u^{(5)}(t) = q(t)f(t, u(t), u^{'}(t), u^{''}(t), u^{'''}(t), u^{(4)}(t)), \quad 0 < t < 1. \tag{1}$$

$$u(0) = u^{'}(1) = u^{''}(0) = u^{'''}(1) = u^{(4)}(0) = 0, \tag{2}$$

where $q : [0, 1] \to [0, \infty)$, $f : [0, 1] \times [0, \infty) \times [0, \infty) \times (-\infty, 0] \times (-\infty, 0] \times [0, \infty) \to [0, \infty)$, are continuous.

This article is organized as follows. In section 2, we present some definitions that will be used to prove the results. Then, in section 3, we present and prove our main results which consists of existence theorems for positive solution of the $(1) - (2)$ without imposing any nonnegativity condition on f. And we establish some existence criteria of at least one positive solution by using the Leray-Schauder nonlinear alternative and Leray-Schauder fixed point theorem. Finally, in section 4, as an application, we give an example to illustrate the results we obtained.

2. Preliminaries

In this section, we present some definitions, Leray-Schauder nonlinear alternative and Leray-Schauder fixed point theorem.

Definition 2.1. *Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone of E if it satisfies the following two conditions*
(1) $x \in P, \lambda > 0$ *implies* $\lambda x \in P$,
(2) $x \in P, -x \in P$ *implies* $x = 0$.

Definition 2.2. *An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.*

Definition 2.3. *Suppose P is a cone in a Banach space E. The map α is a nonnegative continuous concave functional on P provided $\alpha : P \to [0, \infty)$ is continuous and*

$$\alpha(rx + (1-r)y) \geq r\alpha(x) + (1-r)\alpha(y)$$

for all $x, y \in P$ and $r \in [0,1]$. Similarly, we say the map β is a nonnegative continuous convex functional on P provided $\beta : P \to [0, \infty)$ is continuous and

$$\beta(rx + (1-r)y) \leq r\beta(x) + (1-r)\beta(y)$$

for all $x, y \in P$ and $r \in [0,1]$.

We shall use the well-known Leray-Schauder fixed point theorem and Leray-Schauder nonlinear alternative to search for positive solution of the problem $(1)-(2)$.

Theorem 2.4. *([1, 2]). Let E be Banach space and Ω be a bounded open subset of E, $0 \in \Omega$. $T : \overline{\Omega} \to E$ be a completely continuous operator. Then, either*
(i) there exists $u \in \partial\Omega$ and $\lambda > 1$ such that $T(u) = \lambda u$, or
(ii) there exists a fixed point $u^* \in \overline{\Omega}$.

3. Mains results

In this section, we shall impose growth conditions on f, which allow us to apply Leray-Schauder nonlinear alternative, and Leray-Schauder fixed point theorem to establish the existence of at least one positive solution to the $(1)-(2)$, and we assume that $q(t) \equiv 1$.

Lemma 3.1. *Let $E = \{u \in C^4([0,1]) : u(0) = u'(1) = u''(0) = u'''(1) = 0\}$ be the Banach space equipped with the maximum norm*

$$\|u\| = \max\{|u|_0, |u'|_0, |u''|_0, |u'''|_0, |u^{(4)}|_0\},$$

where $|u|_0 = \max_{0\leq t\leq 1} |u(t)|$. Then for any $u \in E$, we have

$$\|u\| = |u^{(4)}|_0 \text{ and } |u|_0 \leq \frac{5}{24}\|u\|,\ |u'|_0 \leq \frac{1}{3}\|u\|,\ |u''|_0 \leq \frac{1}{2}\|u\|,\ |u'''|_0 \leq \|u\|.$$

Proof. Let $G(t,s)$ be the Green's function of fourth-order homogeneous boundary value problem

$$u^{(4)}(t) = 0, \quad 0 < t < 1,$$

with

$$u(0) = u^{'}(1) = u^{''}(0) = u^{'''}(1) = 0.$$

Then

$$G(t,s) = \frac{1}{6}\begin{cases} (6t - 3t^2 - s^2)s, & 0 \le s \le t \le 1, \\ (6s - 3s^2 - t^2)t, & 0 \le t \le s \le 1. \end{cases} \tag{3}$$

By (3) it is easy to know that

$$G(t,s) \ge 0,\ \frac{\partial G(t,s)}{\partial t} \ge 0,\ \frac{\partial^2 G(t,s)}{\partial t^2} \le 0,\ \frac{\partial^3 G(t,s)}{\partial t^3} \le 0, \tag{4}$$

and

$$\int_0^1 |G(t,s)|ds = \int_0^1 G(t,s)ds = \frac{1}{24}t^4 - \frac{1}{6}t^3 + \frac{1}{3}t,$$

$$\int_0^1 |\frac{\partial G(t,s)}{\partial t}|ds = \int_0^1 \frac{\partial G(t,s)}{\partial t}ds = \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{1}{3},$$

$$\int_0^1 |\frac{\partial^2 G(t,s)}{\partial t^2}|ds = -\int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2}ds = -\frac{1}{2}t^2 + t,$$

$$\int_0^1 |\frac{\partial^3 G(t,s)}{\partial t^3}|ds = -\int_0^1 \frac{\partial^3 G(t,s)}{\partial t^3}ds = 1 - t.$$

From which we get

$$\max_{0\le t\le 1}\int_0^1 |G(t,s)|ds = \frac{5}{24},\ \max_{0\le t\le 1}\int_0^1 |\frac{\partial G(t,s)}{\partial t}|ds = \frac{1}{3},$$

$$\max_{0\le t\le 1}\int_0^1 |\frac{\partial^2 G(t,s)}{\partial t^2}|ds = \frac{1}{2},\ \max_{0\le t\le 1}\int_0^1 |\frac{\partial^3 G(t,s)}{\partial t^3}|ds = 1.$$

Let $u \in E$ and $\|u\| = r$. Then

$$u(t) = \int_0^1 G(t,s)[u^{(4)}(s)]ds,\ u^{'}(t) = \int_0^1 \frac{\partial G(t,s)}{\partial t}[u^{(4)}(s)]ds,$$

$$u^{''}(t) = \int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2}[u^{(4)}(s)]ds,\ u^{'''}(t) = \int_0^1 \frac{\partial^3 G(t,s)}{\partial t^3}[u^{(4)}(s)]ds.$$

Thus

$$|u|_0 \le \max_{0\le t\le 1}\int_0^1 |G(t,s)||u^{(4)}(s)|ds \le |u^{(4)}|_0 \max_{0\le t\le 1}\int_0^1 |G(t,s)|ds = \frac{5}{24}|u^{(4)}|_0,$$

$$|u^{'}|_0 \le \max_{0\le t\le 1}\int_0^1 |\frac{\partial G(t,s)}{\partial t}||u^{(4)}(s)|ds \le |u^{(4)}|_0 \max_{0\le t\le 1}\int_0^1 |\frac{\partial G(t,s)}{\partial t}|ds = \frac{1}{3}|u^{(4)}|_0,$$

$$|u^{''}|_0 \le \max_{0\le t\le 1}\int_0^1 |\frac{\partial^2 G(t,s)}{\partial t^2}||u^{(4)}(s)|ds \le |u^{(4)}|_0 \max_{0\le t\le 1}\int_0^1 |\frac{\partial^2 G(t,s)}{\partial t^2}|ds = \frac{1}{2}|u^{(4)}|_0,$$

$$|u^{'''}|_0 \le \max_{0\le t\le 1}\int_0^1 |\frac{\partial^3 G(t,s)}{\partial t^3}||u^{(4)}(s)|ds \le |u^{(4)}|_0 \max_{0\le t\le 1}\int_0^1 |\frac{\partial^3 G(t,s)}{\partial t^3}|ds = |u^{(4)}|_0.$$

So, $|u^{(4)}|_0 = \|u\| = r$ and the proof is completed. □

Theorem 3.2. *Suppose that* $f \in C([0,1] \times [0,\infty) \times [0,\infty) \times (-\infty,0] \times (-\infty,0] \times [0,\infty),[0,\infty))$ *and* $f(t,0,0,0,0,0) \neq 0,\ t \in [0,1]$. *Suppose there exist nonnegative functions* $a_i \in L^1[0,1],\ i = 0,1,2,3,4,5$, *such that*

$$B = \frac{5}{24}\int_0^1 a_0(s)ds + \frac{1}{3}\int_0^1 a_1(s)ds + \frac{1}{2}\int_0^1 a_2(s)ds + \int_0^1 a_3(s)ds + \int_0^1 a_4(s)ds < 1, \tag{5}$$

and for any $(t,u_0,u_1,u_2,u_3,u_4) \in [0,1]\times[0,\frac{5}{24}\rho]\times[0,\frac{1}{3}\rho]\times[-\frac{1}{2}\rho,0]\times[-\rho,0]\times[0,\rho]$, f *satisfies*

$$f(t,u_0,u_1,u_2,u_3,u_4) \leq a_0(t)u_0 + a_1(t)u_1 - a_2(t)u_2 - a_3(t)u_3 + a_4(t)u_4 + a_5(t), \tag{6}$$

where $\rho = A(1-B)^{-1}$, $A = \int_0^1 a_5(s)ds$. *Then problem* $(1)-(2)$ *has at least one positive solution* $u^* \in C^5([0,1])$ *such that*

$$\frac{24}{5}\max_{0\leq t\leq 1} u^*(t) \leq 3\max_{0\leq t\leq 1}(u^*)^{'}(t) \leq 2\max_{0\leq t\leq 1}[-(u^*)^{''}(t)] \leq \max_{0\leq t\leq 1}[-(u^*)^{'''}(t)] \leq \max_{0\leq t\leq 1}(u^*)^{(4)}(t) \leq \rho.$$

Proof. Since $f(t,0,0,0,0,0) \neq 0$ and $|f(t,0,0,0,0,0)| \leq a_5(t),\ t \in [0,1]$, we have $A = \int_0^1 a_5(s)ds > 0$, so, it follows from (3) that $\rho > 0$. From equation (1) and boundary condition $u^{(4)}(0) = 0$ we have

$$u^{(4)}(t) = \int_t^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau))d\tau,$$

which implies that

$$u(t) = \int_0^1 G(t,s)\int_s^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau))d\tau ds,\ t \in [0,1],$$

where $G(t,s)$ is defined by (3). Let $\Omega_\rho = \{u \in E, \|u\| < \rho\}$, then Ω_ρ is a bounded closed convex set of E and $0 \in \Omega_\rho$. For $u \in \Omega_\rho$, define the operator T by

$$(Tu)(t) = \int_0^1 G(t,s)\int_s^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau))d\tau ds. \tag{7}$$

Then

$$(Tu)^{'}(t) = \int_0^1 \frac{\partial G(t,s)}{\partial t}\int_s^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau))d\tau ds,$$

$$(Tu)^{''}(t) = \int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2}\int_s^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau))d\tau ds$$

$$(Tu)^{'''}(t) = \int_0^1 \frac{\partial^3 G(t,s)}{\partial t^3}\int_s^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau))d\tau ds,$$

$$(Tu)^{(4)}(t) = \int_t^1 f(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),u^{'''}(\tau),u^{(4)}(\tau))d\tau, \quad t \in [0,1].$$

So, $(Tu)(0) = (Tu)^{'}(1) = (Tu)^{''}(0) = (Tu)^{'''}(1) = (Tu)^{(4)}(0) = 0$. Therefore, $T : \Omega_\rho \to E$. By Ascoli-Arzela Theorem, it is easy to know that this operator $T:\Omega_\rho \to E$ is a completely continuous operator. So, the problem $(1)-(2)$ has a solution $u=u(t)$ if and only if u solves the operator equation $Tu=u$.

Suppose there exists $u\in\partial\Omega_\rho$, $\lambda>1$ such that $Tu=\lambda u$. Noticing that $\|u\|=\rho$, it follows from lemma 3.1 that

$$|u|_0 \leq \frac{5}{24}\rho,\ |u^{'}|_0 \leq \frac{1}{3}\rho,\ |u^{''}|_0 \leq \frac{1}{2}\rho,\ |u^{'''}|_0 \leq \rho,\ |u^{(4)}|_0 = \rho.$$

Thus from (3), (4) and (5) we have $\lambda\rho=\lambda\|u\|=\|Tu\|=\max_{0\leq t\leq 1}|u^{(4)}(t)|$

$$= \max_{0\leq t\leq 1} |\int_t^1 f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s))ds|$$
$$= \max_{0\leq t\leq 1} \int_t^1 f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s))ds$$
$$= \int_0^1 f(s,u(s),u^{'}(s),u^{''}(s),u^{'''}(s),u^{(4)}(s))ds$$
$$\leq \int_0^1 [a_0(s)u(s)+a_1(s)u^{'}(s)-a_2(s)u^{''}(s)-a_3(s)u^{'''}(s)+a_4(s)u^{(4)}(s)+a_5(s)]ds$$
$$\leq \int_0^1 [\frac{5}{24}a_0(s)\rho+\frac{1}{3}a_1(s)\rho+\frac{1}{2}a_2(s)\rho+a_3(s)\rho+a_4(s)\rho+a_5(s)]ds$$
$$= [\frac{5}{24}\int_0^1 a_0(s)ds+\frac{1}{3}\int_0^1 a_1(s)ds+\frac{1}{2}\int_0^1 a_2(s)ds+\int_0^1 a_3(s)ds+\int_0^1 a_4(s)ds]\rho+\int_0^1 a_5(s)ds$$
$$= B\rho+A = B\rho+(1-B)\rho=\rho,$$

a contradiction. So, by Theorem 2.4, T has a fixed point $u^*\in E$, which is a solution of the problem $(1)-(2)$. Noticing that $f(t,0,0,0,0,0)\neq 0$, we assert that $u=0$ is not a solution of the $(1)-(2)$, therefore, $|u^*|_0>0$. It follows from (3.2) that $u^*(t)$ is nondecreasing and concave on $[0,1]$, thus $u^*(t)\geq t|u^*|_0>0$ for $t\in[0,1]$, i.e., $u^*(t)$ is a positive solution of the problem $(1)-(2)$. This completes the proof. □

4. Example

In order to illustrate the above results, we consider an example.

Example 4.1. *Consider the following problem SBVP*

$$(8)\quad \begin{cases} u^{(5)} = \frac{\sqrt{t}}{26}u + \frac{t^{15}}{2}u^{'} - \frac{t^{11}}{4}u^{''} - \frac{\sqrt[3]{t}}{59}u^{'''} + \frac{t^4}{7}u^{(4)} + t^3 + 1, \\ u(0) = u^{'}(1) = u^{''}(0) = u^{'''}(1) = u^{(4)}(0) = 0. \end{cases}$$

Set

$$f(t,u_0,u_1,u_2,u_3,u_4) = \frac{\sqrt{t}}{26}u_0 + \frac{t^{15}}{2}u_1 - \frac{t^{11}}{4}u_2 - \frac{\sqrt[3]{t}}{59}u_3 + \frac{t^4}{7}u_4 + t^3 + 1,$$

and

$$a_0(t) = \frac{\sqrt{t}}{26},\ a_1(t)=t^{15},\ a_2(t)=\frac{t^{11}}{4},\ a_3(t)=\frac{\sqrt[3]{t}}{59},\ a_4(t)=\frac{t^4}{7},$$
$$a_5(t)=t^3+2.$$

It is easy to prove that $a_i\in L^1[0,1]$, $i=0,1,2,3,4,5$, *are nonnegative functions,* $f(t,0,0,0,0,0)=t^3+1\neq 0$.

Moreover, we have

$$B = \frac{2}{15}\int_0^1 a_0(s)ds + \frac{5}{24}\int_0^1 a_1(s)ds + \frac{1}{3}\int_0^1 a_2(s)ds + \frac{1}{2}\int_0^1 a_3(s)ds + \int_0^1 a_4(s)ds,$$
$$= \frac{2}{15}\int_0^1 \frac{\sqrt{s}}{26}ds + \frac{5}{24}\int_0^1 s^{15}ds + \frac{1}{3}\int_0^1 \frac{s^{11}}{2}ds + \frac{1}{2}\int_0^1 \frac{\sqrt[3]{s}}{59}ds + \int_0^1 \frac{s^4}{7}ds$$
$$= \frac{2}{585} + \frac{5}{384} + \frac{1}{144} + \frac{3}{472} + \frac{1}{35} \simeq 0,056 < 1,$$

and for any

$$(t, u_0, u_1, u_2, u_3, u_4) \in [0,1] \times [0, \frac{5}{24}\rho] \times [0, \frac{1}{3}\rho] \times [-\frac{1}{2}\rho, 0] \times [-\rho, 0] \times [0, \rho],$$

and f satisfies

$$f(t, u_0, u_1, u_2, u_3, u_4) \le a_0(t)u_0 + a_1(t)u_1 - a_2(t)u_2 - a_3(t)u_3 + a_4(t)u_4 + a_5(t).$$

where

$$A = \int_0^1 a_5(s)ds = \frac{9}{4}, \;\; \rho = A(1-B)^{-1} \simeq 2.383.$$

Hence, by theorem 3.2*, the BVP* (8) *has at least one positive solution u^* in* $C^5([0,1])$ *such that*

$$\frac{24}{5}\max_{0\le t\le 1} u^*(t) \le 3\max_{0\le t\le 1}(u^*)'(t) \le 2\max_{0\le t\le 1}[-(u^*)''(t)] \le \max_{0\le t\le 1}[-(u^*)'''(t)] \le$$
$$\max_{0\le t\le 1}(u^*)^{(4)}(t) \le \rho.$$

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ASYMPTOTIC BEHAVIOR OF A NONLINEAR THERMOELASTIC SYSTEM WITH MEMORY TYPE

AMEL BOUDIAF

ABSTRACT. In this work we study a nonlinear system of thermoelasticity, where the viscoelastic dissipation is acting on a part of the boundary, for certain initial data and suitable conditions, we establish a general decay result, from which the usual exponential and polynomial decay are only special cases.

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KEYWORDS AND PHRASES. Thermoelasticity, General decay, Memory, Nonlinear source.

1. INTRODUCTION

In the present work we study the following system

(1)

$$\begin{cases} u_{tt} - \mu \triangle u - (\mu+\lambda)\nabla(divu) + \beta\nabla\theta = |u|^{p-2}u & \text{in } \Omega\times(0,+\infty) \\ c\theta_t - k\triangle\theta + \beta divu_t = 0 & \text{in } \Omega\times(0,+\infty) \\ u(.,0)=u_0,\ u_t(.,0)=u_1,\ \theta(.,0)=\theta_0, & x\in\Omega \\ u(x,t) = -\int_0^t g(t-s)\left(\mu\frac{\partial u}{\partial\nu}+(\mu+\lambda)(divu)\nu\right)(s)\,ds & \text{on } \Gamma_1\times\mathbb{R}^+ \\ u_0=0,\ x\in\Gamma_1,\ \theta=0,\ \ x\in\partial\Omega,\ t\geq 0, & \\ u=0,\ x\in\Gamma_0,\ t\geq 0, & \end{cases}$$

with $c, k, \beta, \mu, \lambda$ are positive constants, where μ, λ are lame moduli, Ω is a bounded domain of $\mathbb{R}^n$, with a smooth boundary $\partial\Omega$, such that $\{\Gamma_0 \cup \Gamma_1\}$ is a partition of $\partial\Omega$, with meas $(\Gamma_1) > 0$, ν is the outward normal to $\partial\Omega$, $u = u(x,t) \in \mathbb{R}^n$ is the displacement vector, $\theta = \theta(x,t)$ is the difference temperature, and g is the relaxation function considered to be positive and of general decay and the boundary condition on Γ_1 is the nonlocal viscoelastic condition responsible for the memory effect. Our aim here is to establish a general decay result, from which the usual exponential and polynomial decay are only special cases.

Preliminaries

In order to establish our result we shall make the following assumption

(H) There exists x_0 in $\mathbb{R}^n$, for which $m(x) = x - x_0$ satisfies

$$m(x).\nu \geq \delta > 0,\ \forall x\in\Gamma_1 \ \text{ and } m(x).\nu \leq 0,\ \forall x\in\Gamma_0.$$

First, we will use the boundary condition

(2)

$$u(x,t) = -\int_0^t g(t-s)\left(\mu\frac{\partial u}{\partial\nu}+(\mu+\lambda)(divu)\nu\right)(s)\,ds,\ x\in\Gamma_1,\ t\geq 0,$$

to estimate the boundary term $\mu\frac{\partial u}{\partial \nu}+(\mu+\lambda)(divu)\nu$. Defining the convolution product operator by

$$(g*\varphi)(t)=\int_0^t g(t-s)\varphi(s)\,ds,$$

and differentiating Eq. (2), we obtain

$$\mu\frac{\partial u}{\partial \nu}+(\mu+\lambda)(divu)\nu+\frac{1}{g(0)}\left(g'*\left(\mu\frac{\partial u}{\partial \nu}+(\mu+\lambda)(divu)\nu\right)\right)=-\frac{1}{g(0)}u_t \quad \text{on } \Gamma_1\times\mathbb{R}^+.$$

Applying Volterra's inverse operator, we get

$$\mu\frac{\partial u}{\partial \nu}+(\mu+\lambda)(divu)\nu=-\frac{1}{g(0)}(u_t+k*u_t) \qquad \text{on } \Gamma_1\times\mathbb{R}^+.$$

Denoting by $\eta=\frac{1}{g(0)}$, we arrive at

$$\mu\frac{\partial u}{\partial \nu}+(\mu+\lambda)(divu)\nu=-\eta\left(u_t+k(0)u+k'*u\right) \qquad \text{on } \Gamma_1\times\mathbb{R}^+. \tag{3}$$

We then define

$$(g\otimes\varphi)(t):=\int_0^t g(t-s)\left|\varphi(t)-\varphi(s)\right|^2 ds, \tag{4}$$

and

$$(g\circ\varphi)(t):=\int_0^t g(t-s)(\varphi(t)-\varphi(s))\,ds. \tag{5}$$

Lemma 1.1. (*Reference* [4]) *If* $g,\ \varphi\in C^1(\mathbb{R}^+)$, *then*

$$(g*\varphi)\varphi_t=-\frac{1}{2}g(t)\left|\varphi(t)\right|^2+\frac{1}{2}g'\otimes\varphi-\frac{1}{2}\frac{d}{dt}\left(g\otimes\varphi-\left(\int_0^t g(s)\,ds\right)\left|\varphi(t)\right|^2\right). \tag{6}$$

2. Decay of solutions

In this section we discuss the asymptotic behavior of the solutions of system (1) when the resolvent kernel k satisfies

$$k(0)>0,\quad k(t)\geq 0,\quad k'(t)\leq 0,\quad k''(t)\geq\gamma(t)\left(-k'(t)\right), \tag{7}$$

where $\gamma:\mathbb{R}^+\to\mathbb{R}^+$ is a function satisfying the following conditions

$$\gamma(t)>0,\quad \gamma'(t)\leq 0,\quad \text{and}\quad \int_0^{+\infty}\gamma(t)\,dt=+\infty. \tag{8}$$

It is clear that γ is decreasing [hence $\gamma'(t)\leq 0$].

By multiplying Eq. $(1)_1$ by u_t and Eq. $(1)_2$ by θ and integrating over Ω, using integration py parts and boundary conditions (3) and (6), one can easily find that the first order energy of system (1) is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\int_\Omega\left[\mu\left|\nabla u\right|^2+\left|u_t\right|^2+(\mu+\lambda)(divu)^2+c\left|\theta\right|^2\right]dx \\ &\quad -\frac{1}{p}\int_\Omega\left|u\right|^p dx-\frac{\eta}{2}\int_{\Gamma_1}k'\otimes u\,d\Gamma_1+\frac{\eta}{2}\int_{\Gamma_1}k(t)\left|u\right|^2 d\Gamma_1. \end{aligned} \tag{9}$$

Remark 1. *By multiplying equation* (1) *by* u_t *and* θ *respectively, integrating over* Ω *and using integration by parts and the boundary condition, we get*

(10)

$$E'(t) \leq -k\int_{\Omega} |\nabla\theta|^2 - \eta\int_{\Gamma_1} |u_t|^2 + \frac{\eta}{2}k'(t)\int_{\Gamma_1} |u|^2 - \frac{\eta}{2}\int_{\Gamma_1}\int_0^t k''(t-s)\,|u(t)-u(s)|^2\,ds \leq 0,$$

for t *in* $[0,T)$. *This means that the energy is uniformly bounded* (*by* $E(0)$) *and is decreasing in* t.

Theorem 2.1. *Given* $(u_0,u_1,\theta_0) \in \left(H_0^1 \times L^2(\Omega) \times H_0^1\right)$. *Assume that* (H) *and* (7) *and* (8) *hold, with*

$$\lim_{t\to\infty} K(t) = 0. \tag{11}$$

Then, for some t_0 *large enough, we have,* $\forall t \geq t_0$.

$$E(t) \leq cE(0)\,e^{-a\int_0^t \gamma(s)ds},$$

where a *is the fixed positive constant and* c *is a generic positive constant.*

The main idea of proof is to construct a Lyapunov functional $L(t)$ equivalent to $E(t)$. To do this we use the multiplier techniques.

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ASYMPTOTIC STABILITY FOR A VISCOELASTIC KIRCHHOFF EQUATION WITH VERY GENERAL TYPE OF RELAXATION FUNCTIONS

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ABSTRACT. In this paper we consider a nonlinear viscoelastic equation with minimal conditions on the $L^1(0,\infty)$ relaxation function g namely $g'(t) \leq -\xi(t) H(g(t))$, where His an increasing and convex function near the origin and ξ is a nonincreasing function. With only these very general assumptions on the behavior of gat infinity, we establish optimal explicit and general energy decay results from which we can recover the optimal exponential and polynomial rates when $H(s) = s^p$ and pcovers the full admissible range $[1, 2)$.

1. PRELIMINARIES

In this work, we consider nonlinear viscoelastic Kirchhoff equation

(1.1)
$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u_{tt} - \left(\xi_0 + \xi_1 \|\nabla u(t)\|^2_{L^2(\Omega)} + \sigma(\nabla u(t), \nabla u_t(t)_{L^2(\Omega)}\right)\Delta u & \\ \qquad + \int\limits_0^t g(t-s)\Delta u(s)ds = u|u|^\gamma & \text{in } \Omega \times \mathbb{R}_+ \\ u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u(x,0) = u_0,\ u_t(x,0) = u_1, & \text{in } \Omega \end{cases}$$

$$M(t) = \xi_0 + \xi_1 \|\nabla u(t)\|^2_{L^2(\Omega)} + \sigma(\nabla u(t), \nabla u_t(t)_{L^2(\Omega)}$$

where

$$M(t) \geq m_0 \qquad \forall t > 0$$

Here Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, ξ_0, ξ_1 and σ are positive constants, $\min\{\rho,\gamma\} > 0$ and $(n-2)\max\{\rho,\gamma\} \leq 2$, the integral term is a finite memory responsible for the viscoelastic damping where g is a positive decreasing function called the relaxation function, and the right hand side of (1.1)1is a source term.

First, we consider the following assumptions

(A)**:** $g : [0,\infty) \to (0,\infty)$is a differentiable function satisfying

$$m_0 - \int\limits_0^{+\infty} g(s)\,ds = l > 0 \tag{1.2}$$

Key words and phrases. nonlinear viscoelastic Kirchhoff equation, stability,energy, General decay.

and there exists a C^1 function $H:(0,\infty)\to(0,\infty)$ which is linear or it is strictly increasing and strictly convex C^2 function on $(0,r]$, $r\leq g(0)$, with $H(0)=H'(0)=0$, such that

$$g'(t)\leq-\zeta(t)H(g(t)),\quad\forall t\geq 0 \tag{1.3}$$

where ξ is a positive nonincreasing differentiable function.

(B)**:** The constants ρ and γ satisfy

$$\min\{\rho,\gamma\}>0\quad\text{and}\quad(n-2)\max\{\rho,\gamma\}\leq 2.$$

We use the standard Lebesgue and Sobolev spaces, with their usual scalar products and norms, and the following Sobolev–Poincaré inequality

$$\|\phi\|_q\leq c_q\|\nabla\phi\|_2,\qquad\phi\in H_0^1(\Omega) \tag{1.4}$$

for $2\leq q\leq 2n/(n-2)$ if $n\geq 3$ or $q\geq 2$ if $n=1,2$. Throughout this paper, c is used to denote a generic positive constant. Now, we introduce the energy functional

$$\begin{aligned}E(t)\ :&=\frac{1}{2+\rho}\int_\Omega\{|u_t|^{\rho+2}\}dx+\frac{1}{2}\left\{\left(\xi_0+\frac{\xi_1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)}-\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}\\&+\frac{1}{2}\int_\Omega\nabla u_t^2(t)dx+\frac{1}{2}(g\Box\nabla u)(t)-\frac{1}{2+\gamma}\int_\Omega|u|^{\gamma+2}\,dx\end{aligned}$$

where

$$(g\Box v)(t)=\int_\Omega\int_0^t g(t-s)|v(t)-v(s)|^2\,dsdx,$$

and the functional

$$J(t)=J(u,u_t):=\left(1-\int_0^t g(s)\,ds\right)\int_\Omega|\nabla u|^2\,dx+\int_\Omega|\nabla u_t|^2\,dx-\int_\Omega|u|^{\gamma+2}\,dx.$$

Proposition 1. *Assume that (A) and (B) hold and $u_0,u_1\in H_0^1(\Omega)$ satisfy*

$$\beta=\frac{c_{\gamma+2}^{\gamma+2}}{l}\left(\frac{2(\gamma+2)}{\gamma l}E(u_0,u_1)\right)^{\frac{\gamma}{2}}<1\quad\textit{and}\quad J(u_0,u_1)>1 \tag{1.5}$$

where $c\gamma+2$ is the best constant in (2.3) with $q=\gamma+2$, then problem (1.1) has a unique global bounded solution satisfying

$$u,u_t\in C\left(\mathbb{R}_+;H_0^1(\Omega)\right),\quad u_{tt}\in L^2\left(\mathbb{R}_+;H_0^1(\Omega)\right)$$

and, for any $t\geq 0$,

$$l\|\nabla u\|_2^2+\|\nabla u_t\|_2^2\leq\frac{2(\gamma+2)}{\gamma}E(0). \tag{1.6}$$

2. **Stability**

In this section we state and prove our main result.

Theorem 1. *Assume that* (A) *and* (B) *hold and* $u_0, u_1 \in H_0^1(\Omega)$ *satisfy* (1.5). *Then there exist positive constants* $k_1 \leq 1$ *and* k_2 *such that, along the solution of* (1.1), *the energy functional satisfies*

$$E(t) \leq k_2 H_1^{-1}\left(k_1 \int_{g^{-1}(r)}^{t} \xi(s)\, ds\right) \quad \text{where} \quad H_1(t) = \int_t^r \frac{1}{sH'(s)} ds. \tag{2.1}$$

Here, H_1 *is strictly decreasing and convex on* $(0, r]$*, with*

$$\lim_{t \to 0} H_1(t) = +\infty.$$

Remark 1. *so, if we define* $H_0(t) = \int_t^r \frac{1}{H(s)} ds$*, then* H_0 *is strictly decreasing and convex on* $(0, r]$*, with* $\lim_{t \to 0} H_0(t) = +\infty$*, and* $H_0(g(t)) \geq \int_{g^{-1}(r)}^{t} \xi(s)\, ds$ *which means*

$$g(t) \leq H_0^{-1}\left(\int_{g^{-1}(r)}^{t} \xi(s)\, ds\right), \qquad \forall t \geq g^{-1}(r).$$

Also, it is evident, by the properties of H*,* H_0 *and* H_1*, that*

$$H_1(t) = \int_t^r \frac{1}{sH'(s)} ds \leq \int_t^r \frac{1}{H(s)} ds = H_0(t) \Longrightarrow H_1^{-1}(t) \leq H_0^{-1}(t)$$

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AN ESTIMATION ON HYPER-ORDER OF SOLUTIONS OF COMPLEX LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS OF SLOW GROWTH

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Abstract. In this paper, we study the growth of meromorphic solutions of higher order linear differential equations with entire coefficients and we obtain some estimations on the hyper-order and hyper convergence exponent of zeros of these solutions. We extend some results due to L. Wang, H. Liu [4] and C. Y. Zhang, J. Tu [5] .

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1. Introduction and main results

Differential equations in the complex domain is an area of mathematics admitting several ways of approach. One of the investigated approach is Nevanlinna's theory. This theory deals with value distribution of meromorphic functions in the complex plane. In this past century, Nevanlinna's theory helped many authors to study the complex differential equations by obtaining many valuable results concerning the growth and oscillation of the solutions of these equations.

As a result, many authors investigated the growth of solutions of the higher order linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), \tag{1}$$

and

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = Qe^P, \tag{2}$$

when $A_j(z)$ $(j = 0, 1, \cdots, k-1)$, $F(z)(\not\equiv 0)$, $Q(z)(\not\equiv 0)$ are entire (or meromorphic) functions and P is a transcendental entire function and obtained some valuable results when there exists some coefficient $A_s(z)$ $(0 \le s \le k-1)$ in equation (1) verifying the condition $\mu(A_s) < \frac{1}{2}$ or when $F(z)$ is of infinite order which is the case in equation (2), (see e.g. [1], [2], [4], [5]).

For $k \ge 2$, we consider the linear differential equation

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), \tag{3}$$

when $A_j(z)$ $(j = 0, 1, \cdots, k)$, $F(z)$ are entire functions such that $A_0A_kF \not\equiv 0$. Many studies showed that if $A_k(z) \equiv 1$, then all solutions of (3) are entire functions, but when $A_k(z)$ is a nonconstant entire function, then equation (3) can possess meromorphic solutions.

For instance the equation

$$zf''' + 4f'' + \left(-1 - \tfrac{1}{2}z^2 - z\right)e^{-z}f' + \left(\left(1 - \tfrac{1}{2}z^2 + 2z\right)e^{-2z} + ze^{-3z}\right)f$$
$$= \left(-1 - \tfrac{1}{2}z^2 - z\right)e^{-z} + \left(z - \tfrac{1}{2}z^3 + 2z^2\right)e^{-2z} + z^2e^{-3z}$$

has a meromorphic solution $f(z) = \frac{1}{z^2}e^{e^{-z}} + z$. Thus, we considered the following questions: firstly, what are the properties of solutions of the linear differential equation (3), when there exists some coefficient $A_s(z)$ $(0 \le s \le k)$ verifying the condition $\mu(A_s) < \frac{1}{2}$? and secondly, how about the growth of meromorphic solutions of the linear differential equation

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = Qe^P, \tag{4}$$

when $A_j(z)$ $(j = 0, 1, \cdots, k)$, $Q(z)(\not\equiv 0)$ are entire functions and P is a transcendental entire function? In this paper, we answered the above questions by obtaining the following results.

Theorem 1.1. *([3]) Suppose that $A_0(z), \cdots, A_k(z)$, $F(z)(\not\equiv 0)$ are entire functions of finite order. If there exists some $s \in \{0, 1, \cdots, k\}$ such that*

$$\alpha = \max\{\sigma(A_j), (j \neq s), \sigma(F)\} < \mu(A_s) < \frac{1}{2},$$

then

(i) Every transcendental meromorphic solution f of (3) such that $\lambda\left(\frac{1}{f}\right) < \mu(f)$, satisfies $\mu(A_s) \le \sigma_2(f) \le \sigma(A_s)$. Furthermore, if $F \not\equiv 0$, then we have $\mu(A_s) \le \bar{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) \le \sigma(A_s)$.

(ii) If $s \ge 2$, then every rational solution f of (3) is a polynomial with $\deg f \le s - 1$. If $s = 0$ or 1, then every nonconstant solution f of (3) is transcendental.

Corollary 1.2. *([3]) Suppose that $A_0(z), \cdots, A_k(z)$, $F(z)(\not\equiv 0)$ are entire functions. If there exists some $s \in \{0, 1, \cdots, k\}$ such that*

$$\alpha = \max\{\sigma(A_j), (j \neq s), \sigma(F)\} < \mu(A_s) = \sigma(A_s) < \frac{1}{2},$$

then every transcendental meromorphic solution f of (3) such that $\lambda\left(\frac{1}{f}\right) < \mu(f)$ satisfies $\bar{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = \sigma(A_s)$, and every rational solution f of (3) is a polynomial with $\deg f \le s - 1$.

Theorem 1.3. *([3]) Suppose that $A_0(z), \cdots, A_k(z)$, $Q(z)(\not\equiv 0)$ are entire functions of finite order, P is a transcendental entire function such that*

$$\max\{\sigma(P), \sigma(Q), \sigma(A_j), (1 \le j \le k)\} < \mu(A_0) < \frac{1}{2}.$$

Then every solution f of (4) is transcendental, and every transcendental meromorphic solution f of (4) such that $\lambda\left(\frac{1}{f}\right) < \mu(f)$ satisfies $\mu(A_0) \le \bar{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) \le \sigma(A_0)$.

Corollary 1.4. *([3]) Suppose that $A_0(z), \cdots, A_k(z)$, $Q(z)(\not\equiv 0)$ are entire functions of finite order, P is a transcendental entire function such that*

$$\max\{\sigma(P), \sigma(Q), \sigma(A_j), (1 \leq j \leq k)\} < \mu(A_0) = \sigma(A_0) < \frac{1}{2}.$$

Then every solution f of (4) is transcendental, and every transcendental meromorphic solution f of (4) such that $\lambda\left(\frac{1}{f}\right) < \mu(f)$ satisfies $\bar{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = \sigma(A_0)$.

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ANALYSE D'UN SYSTÈME DIFFÉRENTIEL FRACTIONNAIRE PERTURBÉ

MOHAMED OMANE, SAFIA MEFTAH, AND LAMINE NISSE

Abstract. The theory of perturbations and the asymptotic analysis concerning ordinary differential equations are widely developed in the mathematical literature. On the other hand, for the fractional case, there are significantly fewer publications in this field of scientific research.
To schematize one considers a problem (of Cauchy or with the limiting values) of the form

$$P_\varepsilon u = 0$$

One of the objectives of this theory is to analyze, and determine the behavior of the solution of the problem when ε tends towards zero.
The work studies the convergence of a solution of a fractional differential system perturbed at Capito sans with initial conditions (a case study of the order perturbation of the fractional derivation and its approach to 1 on the left and right or the order enter 0 and 1.

Keywords and phrases. perturbation,Gronwell theorem,fractional derivatives ,Mittag-Leffler function.

1. Differential problem (having an perturbe fractional order next to the one :

We have the problem :
(p_{ε^-})

$$\begin{cases} ({}^cD^{1-\varepsilon}u_\varepsilon)(t) \;=\; f(t, u_\varepsilon(t)),\; 0 < 1-\varepsilon < 1 \\ u_\varepsilon(0) = \; u_{\varepsilon,0}\;,\; t \in [0,T]\,,\; T < +\infty \end{cases}$$

(p_{ε^+})

$$\begin{cases} ({}^cD^{1+\varepsilon}u_\varepsilon)(t) \;=\; f(t, u_\varepsilon(t)),\; 0 < \varepsilon < 1 \\ u_\varepsilon(0) = \; u_{\varepsilon,0}\;,\; t \in [0,T]\,,\; T < +\infty \\ u_\varepsilon^{'}(0) = \; \varepsilon\mu_0 \end{cases}$$

where : $\|f\| = max_{0\leq t\leq T} \mid f(t,y) \mid= M$, $M \in \mathbb{R}_+$ $\forall y \in \mathbb{R}^n$, let the function f be cuntinuous and fulfill a Lipschitz condition with respect to the second variable ,i.e;

$$\mid f(t,y) - f(t,z) \mid\leq L \mid y - z \mid,\; \forall t \in [0,T]$$

The solution to this problem(p_{ε^-}) is given by relation :

$$U_\varepsilon(t) = u_{\varepsilon,0} + \frac{1}{\Gamma(1-\varepsilon)} \int_0^t (t-s)^{-\varepsilon}\ f(s, u_\varepsilon(s))\ ds$$

The solution to this problem (p_{ε^+})is given by relation :

$$u_\varepsilon(t) = u_{\varepsilon,0} + t\varepsilon\mu_0 + \frac{1}{\Gamma(1+\varepsilon)} \int_0^t (t-s)^{\varepsilon}\ f(s, u_\varepsilon(s))\ ds$$

but the problem (p_0) is of order 1 :

$$\begin{cases} (Du)(t) \ = \ f(t, u(t)), \\ u(0) = \ u_0,\ t \in [0, T] \end{cases}$$

accepts a single solution :

$$u(t) = u_0 + \int_0^t\ f(s, u(s))\ ds$$

What is relationship between :

$$u_\varepsilon(t)\ and\ u(t)\ ?\ when\ \varepsilon \longrightarrow 0\ and\ t \in [0, T]$$

$u_\varepsilon(t)$:solution of Differential problem the same fractional derivative are perturbe order et $u(t)$:solution of a non perturbe Differential problem of the first order

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Analytic Gevrey well-posedness and regularity for class of coupled periodic KdV systems of Majda-Biello type

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Keywords:Coupled periodic KdV systems, Well-posedness, Analytic Gevrey spaces, Bourgain spaces,local well-posedness, Majda-Biello, Regularity.

Our purpose in this article is to study the well-posedness and regularity for the coupled Kotewege-de Varie (KdV) system.
We proved the local well-posedness in analytic Gevrey spaced and we studied the Gevrey's regularity in time variable.

$$\begin{cases} u_t + u_{xxx} + ww_x = 0 \\ w_t + \beta w_{xxx} + (uw)_x = 0, \quad x \in \mathbb{T}_\gamma \ , t \in \mathbb{R}, \ 0 < \beta < 1 \\ (u, w) \mid_{t=0} = (u_0, w_0). \end{cases} \tag{1}$$

where $\mathbb{T}_\gamma = [0, 2\pi\gamma)$ for some $\gamma \geq 1$.

The analytic Gevrey spaces with $\gamma \geq 1$ are given by $\mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma) = \mathcal{G}_{\sigma,\delta,s}$. For $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, let us define

$$\mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma) = \left\{ f \in L^2(\mathbb{T}_\gamma); \|f\|^2_{\mathcal{G}^{\sigma,\delta,s}(\mathbb{T}_\gamma)} = \sum_{k\in\mathbb{Z}} e^{2\delta|k|^{1/\sigma}} \langle k \rangle^{2s} |\widehat{f}(k)|^2 d\xi < \infty \right\}, \tag{2}$$

where $\langle \cdot \rangle = (1 + |\cdot|)$.
At a time, the analytic Gevrey-Bourgain spaces $X^\beta_{\sigma,\delta,s,b}(\mathbb{T}_\gamma \times \mathbb{R}) = X^\beta_{\sigma,\delta,s,b}$ and $X_{\sigma,\delta,s,b}(\mathbb{T}_\gamma \times \mathbb{R}) = X_{\sigma,\delta,s,b}$ are defined by

$$\|u\|_{X_{\sigma,\delta,s,b}(\mathbb{T}_\gamma\times\mathbb{R})} = \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{2\delta|k|^{1/\sigma}} \langle k \rangle^{2s} \left\langle \tau - k^3 \right\rangle^{2b} \mid \widehat{u}(k,\tau) \mid^2 d\tau \right)^{\frac{1}{2}}, \tag{3}$$

$$\|w\|_{X^\beta_{\sigma,\delta,s,b}(\mathbb{T}_\gamma\times\mathbb{R})} = \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{2\delta|k|^{1/\sigma}} \langle k \rangle^{2s} \left\langle \tau - \beta k^3 \right\rangle^{2b} \mid \widehat{w}(k,\tau) \mid^2 d\tau \right)^{\frac{1}{2}}. \tag{4}$$

1

The proof of local well-posedness is based on the iteration in the spaces $X_{\sigma,\delta,s,1/2} \times X^{\beta}_{\sigma,\delta,s,1/2}$.and the spaces $Y_{\sigma,\delta,s}(\mathbb{T}_\gamma \times \mathbb{R}) = Y_{\sigma,\delta,s}$ and $Y^{\beta}_{\sigma,\delta,s}(\mathbb{T}_\gamma \times \mathbb{R}) = Y^{\beta}_{\sigma,\delta,s}$ defined via the norms

$$\|u\|_{Y_{\sigma,\delta,s}} = \|u\|_{X_{\sigma,\delta,s,1/2}} + \|e^{\delta|k|^{1/\sigma}} \langle k\rangle^s \widehat{u}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})} \tag{5}$$

and

$$\|w\|_{Y^{\beta}_{\sigma,\delta,s}} = \|w\|_{X^{\beta}_{\sigma,\delta,s,1/2}} + \|e^{\delta|k|^{1/\sigma}} \langle k\rangle^s \widehat{w}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})} \tag{6}$$

For any interval $I \subset \mathbb{R}$, we define the localized spaces $Y^{I}_{\sigma,\delta,s} = Y_{\sigma,\delta,s}(\mathbb{T}_\gamma \times I)$ and $Y^{\beta,I}_{\sigma,\delta,s} = Y^{\beta}_{\sigma,\delta,s}(\mathbb{T}_\gamma \times I)$ with the norms

$$\|u\|_{Y^{I}_{\sigma,\delta,s}} = inf\left\{\|U\|_{Y_{\sigma,\delta,s}}; U|_{(\mathbb{T}_\gamma\times I)} = u\right\} \tag{7}$$

and

$$\|w\|_{Y^{\beta,I}_{\sigma,\delta,s}} = inf\left\{\|W\|_{Y^{\sigma,\beta}_{\delta,s}}; W|_{(\mathbb{T}_\gamma\times I)} = w\right\} \tag{8}$$

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APPLICAION OF HOMOTOPY ANALYSIS METHOD TO THE VARIABLE COEFFICIENT KDV-BURGERS EQUATION

AHCENE BOUKEHILA

Abstract. The paper presents an application of the homotopy analysis method for solving the Variable Coefficient KdV-Burgers Equation. In this method a series is created, sum of which (if the series is convergent) gives the solution of discussed equation. Conditions ensuring convergence of this series are presented in the paper. Error of approximate solution, obtained by considering only partial sum of the series, is also estimated. Its validity is verified by comparing the approximation series with the known exact solution. And different from perturbation techniques, this approach is independent upon any small/large perturbation quantities. So, the basic ideas of this approach can be employed to search for multiple solutions of strongly nonlinear problems in science and engineering.

2010 Mathematics Subject Classification. 65R20 45G10, 45A05.

Keywords and phrases. Homotopy analysis method, KdV-Burgers Equation, Convergence.

1. Statement of the problem

The Gelfand equation represents the steady state of diffusion and transfer of heat conduction see [1]. In this paper, based on the homotopy analysis method [2], a new approach is proposed to solve multiple solutions of strongly nonlinear problems by using Gelfand equation

$$\begin{cases} \triangle u + \lambda e^u, x \in [0,1], \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

Note that the Gelfand equation contains an exponent term $exp(u)$ and thus has very strong nonlinearity. And different from perturbation techniques, this approach is independent upon any small/large perturbation quantities. So, the basic ideas of this approach can be employed to search for multiple solutions of strongly nonlinear problems in science and engineering.

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR A VISCOELASTIC EQUATION WITH NONLINEAR BOUNDARY DAMPING AND SOURCE TERMS

BILLEL GHERAIBIA AND NOURI BOUMAZA

Abstract. In this paper, we consider the initial boundary value problem for the viscoelastic equation with nonlinear boundary damping and source terms. Under suitable assumptions on the relaxation function, we will concern the global existence, general decay, and blow-up result of solutions.

2010 Mathematics Subject Classification. 35L70, 35B40, 35B35.

Keywords and phrases. Viscoelastic equation, Nonlinear boundary conditions, Global existence, General decay, blow-up.

1. Define the problem

In this paper, we study the initial boundary value problem for the following viscoelastic equation with nonlinear boundary damping and source terms

$$\begin{cases} u_{tt} - \operatorname{div}[a(x)\nabla u] + \displaystyle\int_0^t g(t-s)\operatorname{div}[a(x)\nabla u(s)]ds = 0, & \Omega \times (0,+\infty) \\ u = 0, & \Gamma_0 \times (0,+\infty) \\ a(x)\dfrac{\partial u}{\partial v} - a(x)\displaystyle\int_0^t g(t-s)\dfrac{\partial u}{\partial v}ds + |u_t|^{m-2}u_t = |u|^{p-2}u, & \Gamma_1 \times (0,+\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n\,(n \geq 1),\ \partial\Omega = \Gamma_0 \cup \Gamma_1,\ \operatorname{mes}(\Gamma_0) > 0, \Gamma_0 \cap \Gamma_1 = \emptyset,\ \frac{\partial u}{\partial \nu}$ denotes the unit outer normal derivative, p, $m > 2$, $a(x)$ and $g(t)$ are positive functions, and u_0, u_1 are given functions belonging to suitable spaces.

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BLOW UP OF SOLUTIONS FOR A HYPERBOLIC-TYPE EQUATION WITH DELAY TERM AND LOGARITHMIC NONLINEARITY

HAZAL YÜKSEKKAYA AND ERHAN PİŞKİN

Abstract. In this paper, we consider a hyperbolic-type equation with delay term and logarithmic nonlinearity in a bounded domain. Under suitable conditions, we prove the blow up of solutions in a finite time. Generally, time delay effects arise in many applications and practical problems such as physical, chemical, biological, thermal and economic phenomena. We study more general version of the equation:

$$u_{tt} - u_{xx} + u - \varepsilon u \log |u|^2 = 0.$$

2010 Mathematics Subject Classification. 35B05, 35B44, 35L05.

Keywords and phrases. Blow up, Delay term, Hyperbolic equation.

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BLOW-UP, EXPONENTIAL GROWTH OF SOLUTION FOR A NONLINEAR PARABOLIC EQUATION WITH $p(x)$ –LAPLACIAN

AMAR OUAOUA

Abstract. **In this paper, we consider the following equation**

$$u_t - \text{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + \omega |u|^{m(x)-2} u_t = b|u|^{r(x)-2} u. \tag{1}$$

We prove a finite time blowup solutions in the case $\omega = 0$, and exponential growth in the case $\omega > 0$, with the negative initial energy in the both cases.

2010 Mathematics Subject Classification. 35B40; 35L90.

Keywords and phrases. **Nonlinear parabolic equation, $p(x)$ –Laplacian, Blow-up, Exponential grouth.**

1. Define the problem

Equation (1) can be viewed as a generalization of the evolutional p-Laplacian equation

$$u_t - \text{div}\left(|\nabla u|^{p-2}\nabla u\right) + \omega |u|^{m-2} u_t = b|u|^{r-2} u, \tag{2}$$

with the constant exponent $p,\ m,\ r \in (2,\ \infty)$, which appears in various physical contexts. In particular, this equation arises from the mathematical description of the reaction-diffusion/ diffusion, heat transfer, population dynamics processus, and so on (*see* [11]) and references therein). Recently in [1], in the case $\omega = 0$, Agaki proved an existence and blow up result for the initial datum $u_0 \in L^r(\Omega)$. Ôtani [17] studied the existence and the asymptotic behavior of solutions of (2) and overcome the difficulties caused by the use of nonmonotone perturbation theory. The quasilinear case, with $p \neq 2$, requires a strong restriction on the growth of the forcing term $|u|^{r-2}u$, which is caused by the loss of the elliptic estimate for the $p-$Laplacian operator defined by $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ (*see* [8]).

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Bounded and unbounded positive solutions for singular ϕ-Laplacian BVPs on the half-line with first-order derivative dependence

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Abstract

In this talk, we present existence results for positive solutions to the singular ϕ-Lpalcian boundary value problem

$$\begin{cases} -(\phi(u'))' = a(t)f(t,u,u'), \ t \in (0,+\infty) \\ u(0) = \lim\limits_{t\to+\infty} u'(t) = 0, \end{cases}$$

where $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, $a\colon (0,+\infty) \to \mathbb{R}^+$ is a measurable function with $a(t) > 0$ a.e. t in some interval of $(0,+\infty)$ and the nonlinearity $f\colon \mathbb{R}^+ \times (0,+\infty) \times (0,+\infty) \to \mathbb{R}^+$ is continuous and may exhibit singular at the solution and at its derivative.

Key words: ϕ-Laplacian, positive solution, singular boundary value problem.
2010 Mathematics Subject Classification: 34B15, 34B16, 35B18, 34B40.

1 Introduction and main results

This talk concerns existence of positive solutions to the second order boundary value problem (bvp for short)

$$\begin{cases} -\left(\phi(u')\right)'(t) = a(t)f(t,u(t),u'(t)) \text{ a.e. } t > 0, \\ u(0) = \lim\limits_{t\to+\infty} u'(t) = 0, \end{cases} \tag{1.1}$$

where $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, $a\colon (0,+\infty) \to \mathbb{R}^+$ is a measurable function with $a(t) > 0$ a.e. t in some interval of $(0,+\infty)$ and the nonlinearity $f\colon \mathbb{R}^+ \times (0,+\infty)^2 \to \mathbb{R}^+$ is continuous and may exhibit singular at $u = 0$ and $u' = 0$.

By a positive solution to the bvp (1.1), we mean a function u in $C^1([0,+\infty), \mathbb{R})$ such that $u > 0$ in $(0,+\infty)$ and $\phi(u')$ is absolutely continuous on compact intervals of $[0,+\infty)$, satisfying all equations in (1.1).

Our approach in this talk is based on a fixed point formulation and since the weight a and the nonlinearity f will supposed to be nonnegative functions, we will use in this work an adapted version of the Guo-Krasnoselskii's expansion and compression of a cone principal. Because of the singular nature of the nonlinearity f as well as its dependance on the first derivative and the boundary conditions in (1.1), we look for solutions in the cone of nonnegative and concave function belonging to the linear space E of all fonctions $u \in C^1([0,+\infty))$, satisfying $u(0) = \lim_{t\to\to+\infty} u'(t) = 0.$

Notice that functions u in E can be bounded, such is the case for $u_0(t) = \frac{t}{1+t}$, or unbounded as $u_1(t) = \ln(1+t)$. we provide in this talk conditions which guarantee the boundedness or the unboundedness of the obtained solution. In the following, we set $\psi :\ = \phi^{-1}$ and we suppose that a, ϕ and f satisfy the following conditions:

$$\begin{cases} \text{there exists } \alpha > 0 \text{ such that for all } t \in [0,1] \\ \text{and } u \in \mathbb{R}^+, \quad \phi(tu) \geq t^\alpha \phi(u), \end{cases} \tag{1.2}$$

$$|a|_1 = \int_0^{+\infty} a(s)ds < \infty; \tag{1.3}$$

$$\begin{cases} \text{For all } R > 0 \text{ there exists a nonincreasing function} \\ \psi_R : (0,+\infty) \to (0,+\infty) \text{ such that} \\ f(t,(1+t)w,z) \leqslant \psi_R(w) \text{ for all } t,w,z \geq 0 \text{ with } w \leq R \\ \text{and } \int_0^{+\infty} a(t)\psi_R(r\widetilde{\rho}(t))\,dt < \infty \text{ for all } r \in (0,R]. \end{cases} \tag{1.4}$$

where

$$\widetilde{\rho}(t) = \frac{\rho(t)}{1+t} \quad \text{and} \quad \rho(t) = \begin{cases} t \text{ if } t \in [0,1] \\ \frac{1}{t} \text{ if } t \geq 1, \end{cases}$$

$$\begin{cases} \lim_{t\to+\infty} t\psi\left(\int_t^{+\infty} a(s)f(s,\lambda,\mu)ds\right) = +\infty \\ \text{uniformly for } \lambda,\ \mu \text{ in compact intervals of } (0,+\infty), \end{cases} \tag{1.5}$$

$$\begin{cases} \text{For all } R > 0 \text{ there exists a function } \phi_R : (0,+\infty) \to (0,+\infty) \\ \text{such that } f(t,(1+t)w,z) \leqslant \phi_R(t) \text{ for all } t,w,z \geq 0 \text{ with } w \leq R \\ \text{and } \int_0^{+\infty} \psi\left(\int_s^{+\infty} a(r)\phi_R(r)\,dr\right) < \infty. \end{cases} \tag{1.6}$$

The statement of the main result in this talk needs to introduce the following notations. Let $\theta > 1$ be fixed and set $I_\theta = [1/\theta, \theta]$,

$$f^0 = \limsup_{|(w,z)|\to 0}\left(\sup_{t\geq 0}\frac{f(t,(1+t)w,z)}{\phi(w+z)}\right), \qquad f^\infty = \limsup_{|(w,z)|\to+\infty}\left(\sup_{t\geq 0}\frac{f(t,(1+t)w,z)}{\phi(w+z)}\right),$$

$$f_0(\theta) = \liminf_{|(w,z)|\to 0}\left(\min_{t\in I_\theta}\frac{f(t,(1+t)w,z)}{\phi(w)}\right), \quad f_\infty(\theta) = \liminf_{|(w,z)|\to+\infty}\left(\min_{t\in I_\theta}\frac{f(t,(1+t)w,z)}{\phi(w)}\right),$$

$$\Gamma = \left(2^\alpha |a|_1\right)^{-1}, \qquad \Theta(\theta) = (1+\theta)^{2\alpha}(\theta)^\alpha\left(\int_{\frac{1}{\theta}}^{\theta} a(r)dr\right)^{-1}.$$

where $|(w,z)| = \sup(|w|,|z|)$.

Theorem 1.1 *Assume that Hypotheses (1.2)-(1.4) hold and there exists $\theta > 1$ such that one of the following conditions*

$$f^0 < \Gamma, \quad \Theta(\theta) < f_\infty(\theta) \tag{1.7}$$

and

$$f^\infty < \Gamma, \quad \Theta(\theta) < f_0(\theta) \tag{1.8}$$

is satisfied. Then bvp (1.1) has at least one positive solution u. Moreover, if Hypothesis (1.6) holds then the solution u is bounded and if Hypothesis (1.5) holds, then the solution u is unbounded (i.e. $\lim\limits_{t\to+\infty} u(t) = +\infty$).

Since for all $t, w, z > 0$

$$\frac{f(t,(1+t)\,w,z)}{\phi(w+z)} \le \frac{f(t,(1+t)\,w,z)}{\phi(w)}$$

we have

$$f^0 \le f_+^0 = \limsup_{w\to 0}\left(\sup_{t,z>0}\frac{f(t,(1+t)w,z)}{\phi(w+z)}\right), \qquad f_0(\theta) \ge f_0^-(\theta) = \liminf_{w\to 0}\left(\inf_{t\in I_\theta, z>0}\frac{f(t,(1+t)w,z)}{\phi(w)}\right),$$
$$f^\infty \le f_+^\infty = \limsup_{w\to+\infty}\left(\sup_{t,z>0}\frac{f(t,(1+t)w,z)}{\phi(w+z)}\right), \quad f_\infty(\theta) \ge f_\infty^-(\theta) = \liminf_{w\to+\infty}\left(\inf_{t\in I_\theta, z>0}\frac{f(t,(1+t)w,z)}{\phi(w)}\right),$$

Moreover, notice that if the following hypothesis

$$\begin{cases} \text{for all } R>0 \text{ there exists a function } \omega_R:(0,+\infty)\to(0,+\infty) \\ \text{such that } f(t,u,v)\ge\omega_R(t) \text{ for all } t,u,v>0 \text{ with } u\le R \\ \text{and } \lim_{t\to+\infty} t\psi\left(\int_t^{+\infty} a(r)\omega_R(r)\,dr\right) = +\infty, \end{cases} \tag{1.9}$$

holds, then the nonlinearity f satisfies (1.5).

The above remarks and Theorem 1.1 lead to the following corollary:

Corollary 1.2 *Assume that Hypotheses (1.2)-(1.4) hold and there exists $\theta > 1$ such that one of the following conditions*

$$f_+^0 < \Gamma, \quad \Theta(\theta) < f_\infty^-(\theta)$$

and

$$f_+^\infty < \Gamma, \quad \Theta(\theta) < f_0^-(\theta)$$

is satisfied. Then the bvp (1.1) has at least one positive solution u. Moreover, if Hypothesis (1.6) holds then the solution u is bounded and if Hypothesis (1.9) holds, then the solution u is unbounded.

2 Abstract Background

Let X be a linear space and let $\|\cdot\|_N$ and p be respectively a norm and a semi-norm on X such that $(X, \|\cdot\|)$ is a Banach space, where for $x \in X$, $\|x\| = \max(\|x\|_N, p(x))$. Let K be a cone in X, that is: K is nonempty closed and covex such that $K \cap (-K) = \emptyset$ and $tK \subset K$ for all $t \ge 0$. The main result of this work will be proved by means of the following theorem:

Theorem 2.1 ([9], Theorem 2.8) *Let r_1, r_2 be two positive real numbers such that $r_1 < r_2$ and let $T\colon K\cap(\overline{\Omega_2}\setminus\Omega_1) \to K$ be a compact mapping where for $i = 1, 2$, $\Omega_i = \{u \in E, \|u\|_N < r_i\}$. If one of the following conditions*
(a) $\|Tu\| \le \|u\|$ for $u \in K\cap\partial\Omega_1$ and $\|Tu\|_N \ge \|u\|_N$ for $u \in K\cap\partial\Omega_2$,
(b) $\|Tu\|_N \ge \|u\|_N$ for $u \in K\cap\partial\Omega_1$ and $\|Tu\| \le \|u\|$ for $u \in K\cap\partial\Omega_2$.
is satisfied, then T has at least a fixed point in $K \cap (\Omega_2 \setminus \bar{\Omega_1})$.

The above theorem is a new version of expansion and compression of a cone principal in a Banach space. Its improvement consists in the fact that it does not require bounded sets.

3 Example

Consider the bvp (1.1) in the case where

$$\begin{aligned} \phi(x) &= |x|^{p-2}\,x + |x|^{q-2}\,x, && 1 < p < q \\ a(t) &= \frac{1}{(1+t)^{\xi}}, && \xi > 1 \end{aligned}$$

and

$$f(t,u,v) = \left(\left(\frac{Au}{1+t}\right)^{m-1} + \left(\frac{Bu}{1+t}\right)^{n-1}\right)\left(2 + \frac{z}{1+z} + \sin\left(\frac{1+t}{u} + \frac{1}{v}\right)\right)$$

with $A, B > 0$, $m < n$ and $n > 1$.

Using Corollary 1.2, bvp (1.1) admits a positive solutions in suitable siuations.

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Closed range positif operators on Hilbert spaces

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Abstract

Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. The reduced minimum modulus of an operator $S \in B(H)$ is defined by

$$\gamma(S) := \begin{cases} \inf\{||Sx||; ||x|| = 1,\, x \in \mathcal{N}(S)^{\perp}\} & if\ S \neq 0 \\ +\infty & if\ S = 0. \end{cases}$$

In this paper, we show that if S is a positif operator and $\alpha > 0$, then $R(S)$ is closed if and only if $\gamma(S^{\alpha}) > 0$. In this case $\mathcal{R}(S) = \mathcal{R}(S^{\alpha})$. Also in this paper, We study the Moore-Penrose inverse of a positif operator.

Keywords: Closed range operator, The reduced minimum modulus, Moore-Penrose inverse.

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CONTROLLABILITY OF DELAY FRACTIONAL SYSTEMS

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Abstract. In this work, some sufficient and necessary conditions the complete controllable of fractional linear system with delays in the state are established. Further, the complete controllability result for a semi-linear fractional system with delays in the state are studied by using Krasnoselskii's fixed point theorem. Finally, numerical examples are given to illustrate our results.

Keywords and phrases. fractional systems, controllability, delays in the state.

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DERIVATION RANGE AND THE IDENTITY OPERATOR

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Abstract. The main objective of this paper is to present some results about classes of operators where the distance between the identity operator and the derivation range is minimal or maximal.

2010 Mathematics Subject Classification. 47B47, 47B20, 47A25.

Keywords and phrases. Range of derivation, identity operator, finite operator, reduced spectrum.

Abstract

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex and infinite dimensional Hilbert space $\mathcal{H}$. The additive mapping $\delta_{A,B} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by $\delta_{A,B}(X) = AX - XB$, for all A, B, $X \in \mathcal{B}(\mathcal{H})$ is called generalized derivation associated with (A, B). If $A = B$, then $\delta_{A,A} = \delta_A$ is called the inner derivation implemented by $A \in \mathcal{B}(\mathcal{H})$.

It is known that the identity operator I is not a commutator, i.e. $I \notin R(\delta_A)$ for any $A \in \mathcal{B}(\mathcal{H})$, where $R(\delta_A)$ denotes the range of δ_A. However, J. H. Anderson [1] showed that there are operators A for which $I \in \overline{R(\delta_A)}$, where $\overline{R(\delta_A)}$ is the closure of $R(\delta_A)$ in the norm topology. This allowed him to define a new class of operators called:

$$\begin{aligned} \mathcal{JA}(\mathcal{H}) &= \{A \in \mathcal{B}(\mathcal{H}) : I \in \overline{R(\delta_A)}\} \\ &= \{A \in \mathcal{B}(\mathcal{H});\ \exists (X_n) \in \mathcal{B}(\mathcal{H}) : AX_n - X_nA \to I\}, \end{aligned}$$

which is the class of operators where the distance between the identity operator and the derivation range is minimal.

The class of operators A in $\mathcal{B}(\mathcal{H})$, where the distance between the inner derivation range $R(\delta_A)$ and the identity operator I is maximal, is called finite operators class and noted by $\mathcal{F}(\mathcal{H})$. In other words, $A \in \mathcal{B}(\mathcal{H})$ is a finite operator if :

$$\|AX - XA - I\| \geq 1;\ \forall\ X \in \mathcal{B}(\mathcal{H}).$$

The purpose of this work is to prove that $\mathcal{JA}(\mathcal{H})$ have not an algebraic structure and that $\mathcal{F}(\mathcal{H})$ is a field, give some sufficient and necessary conditions that the identity operator be in the closure of the range of an inner derivation and to present some results concerning the form of operators in $\mathcal{JA}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H})$.

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POSITIVE PERIODIC SOLUTIONS FOR AN ITERATIVE HEMATOPOIESIS MODEL

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Abstract. In this work, an iterative hematopoiesis model is investigated. By utilizing Schauder's fixed point theorem and some properties of a Green's function, we establish some new existence results about positive periodic solutions of the model. Our main results which improve and generalize the past literature.

2010 Mathematics Subject Classification. 34K13, 34A34, 34C60.

Keywords and phrases. Haematopoiesis model, positive periodic solution, Schauder's fixed point theorem.

1. The problem

This paper deals with the existence of periodic solutions of the following iterative hematopoiesis model with periodic coefficients:

$$x'(t) = -a(t)\,x(t) + p(t)\sum_{i=1}^{n} \frac{x^m(t-\tau(t))}{1+x^{[i]}(t)}, \tag{1}$$

where $m \geq 0$, $x^{[n]}(t)$ is the $n-$th iterate of $x(t)$ and a, p are continuous periodic functions on $\mathbb{R}^+$.

Equation (1) describes the dynamics of hematopoiesis (blood cell production) where $x(t)$ is the density of mature cells in blood circulation at time t, $a(t)$ is the destruction rate, $p(t)$ is the maximal production rate, $\sum_{i=1}^{n} \frac{p(t)\,x^m(t-\tau(t))}{1+x^{[i]}(t)}$ denotes the flux of the cells into the circulation from the stem cell compartment which involves two type of delays, a time-varying delay $\tau(t)$ representing the average of cell cycles and multiple implicit delays of the form $\tau_i(t, x(t))$ depending on both the time and the state variable and representing times required to produce mature cells.

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CONTINUITY OF PSEUDO-DIFFERENTIAL OPERATORS ON LOCALIZED BESOV-TYPE SPACES

AISSA DJERIOU

Abstract. We will study the continuity of some pseudo-differential operators on the localized Besov–type spaces $\left(B^{s,\tau}_{p,q}(\mathbb{R}^n)\right)_{\ell^r}$, under some conditions on s and τ.

2010 Mathematics Subject Classification. Pseudo-differential operators, Besov–type spaces, localized Besov–type spaces.

Keywords and phrases.47E38, 47G30.

1. Introduction and the main result

In this work, we will be interested by the Besov–type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$, is the set of $f \in \mathcal{S}'$, such that

$$\|f\|^q_{B^{s,\tau}_{p,q}} = \sup_{x\in\mathbb{R}^n, J\in\mathbb{Z}} \frac{1}{|B(x,2^{-J})|^{\tau/q}} \sum_{j\geq J} (2^{jsq} \left\|\mathcal{F}^{-1}\varphi_j \cdot \widehat{f}\right\|^q_{L^p(B_J(x,2^{-J}))},$$

where $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is the smooth dyadic resolution of unity in $\mathbb{R}^n$.

We denote by $(B^{s,\tau}_{p,q}(\mathbb{R}^n))_{\ell^r}$ the localized of Besov–type space , Let $\beta \in C^\infty$ where $\operatorname{supp}\beta \subset B(0,Q)$ with $Q > \sqrt{n}$, and satisfying

$$\sum_{j\in\mathbb{Z}^n} \beta(x-j) = 1, \qquad (\forall x \in \mathbb{R}^n). \tag{1}$$

Let $1 \leq p, q, r \leq \infty$. The space $(B^{s,\tau}_{p,q}(\mathbb{R}^n))_{\ell^r}$ is the collection of all $f \in \mathcal{S}'$, such that

$$\|f\|_{(B^{v,\tau}_{p,q})_{\ell^r}} := \Big(\sum_{k\in\mathbb{Z}^n} \|\tau_k\psi \cdot f\|^r_{B^{s,\tau}_{p,q}}\Big)^{1/r},$$

We will study on $(B^{s,\tau}_{p,q}(\mathbb{R}^n))_{\ell^r}$ the continuity of (ps.d.o.) $\sigma(x,D)$ which is defined by

$$\sigma(x,D)f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\mathrm{i}x\cdot\xi} \sigma(x,\xi)\, \widehat{f}(\xi)\, d\xi\,,$$

where σ is a complex-valued and sufficiently differentiable function defined on $\mathbb{R}^n \times \mathbb{R}^n$. The general literature concerning the continuity of ps.d.o on Besov space $B^s_{p,q}(\mathbb{R}^n)$, or on Triebel–Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$, can be found in the different works, as [4]. We recall the set $S^m_{1,\delta}(\omega, N)$ of the type Hörmander class, which presents basic tool in the theory of ps.d.o, see e.g. [1]. Let the function:

$$\sigma_{\alpha,\beta}(x,\xi) := (1+|\xi|)^{|\alpha|-\delta|\beta|-m} \partial^\alpha_\xi \partial^\beta_x \sigma(x,\xi),$$

where $\alpha, \beta \in \mathbb{N}^n$ with $|\beta| \leq N$. $\sigma_{\alpha,\beta}$ is satisfies, for all $x, \xi \in \mathbb{R}^n$, the estimates

$$|\sigma_{\alpha,\beta}(x,\xi)| \leq c_1 \,, \tag{2}$$

$$|\sigma_{\alpha,\beta}(x+h,\xi) - \sigma_{\alpha,\beta}(x,\xi)| \leq c_2\, \omega(|h|\, |\xi|^{\delta}) \,, \tag{3}$$

where ω is a positive nondecreasing function, vanishing near the origin and concave on $\mathbb{R}^+$ (a *modulus of continuity*). The set $S^m_{1,\delta}\,(\omega, N)$ is the collection of such σ.

The main result of this paper has been proved in the case of localized Besov space $(B^s_{p,q}\,(\mathbb{R}^n))_{\ell^r}$ by Moussai [2] and in the case of generalized Triebel–Lizorkin space $F^{v_\mu}_{p,q}\,(\mathbb{R}^n)$ by Djeriou-Moussai [3].

For brevity, throughout this paper some parameters are fixed in the following way, $a > 0,\ \tau \geq 0,\ m \geq 0,\ N \in \mathbb{N},\ 1 < p < \infty,\ 1 < q \leq \infty$ and $0 \leq \delta < 1$, except if they are mentioned in another form. Therefore, we will prove that the following condition of Dini's type:

$$\Big(\sum_{j=1}^{\infty}\Big(2^{-(1-\delta+s)jN}\,\omega\big(2^{-(1-\delta)j}\big)\Big)^q\Big)^{1/q} < +\infty \tag{4}$$

is sufficient and optimal for $(B^{s,\tau}_{p,q})_{\ell^r}-$ continuity. Then our contribution is the following results

Theorem 1.1. *Let $s > N$. Suppose (4). Then every ps.d.o $\sigma\,(x, D)$ of symbol $\sigma \in S^m_{1,\delta}\,(\omega, N)$ is a bounded from $(B^{s+m,\tau}_{p,q}\,(\ \mathbb{R}^n))_{\ell^r}$ to $(B^{s,\tau}_{p,q}\,(\mathbb{R}^n))_{\ell^r}$.*

Theorem 1.2. *Suppose*

$$\Big(\sum_{j=1}^{\infty}\Big(2^{-(1-\delta+s)jN}\,\omega\big(2^{-(1-\delta)j}\big)\Big)^q\Big)^{1/q} = +\infty \tag{5}$$

Then there exist a ps.d.o $\sigma\,(x, D)$ of symbol $\sigma \in S^m_{1,\delta}\,(\omega, N)$ and a function $h \in (B^{s+m,\tau}_{p,q}\,(\mathbb{R}^n))_{\ell^r}$ such that $\sigma\,(x, D)\, h \notin (B^{s,\tau}_{p,q}\,(\mathbb{R}^n))_{\ell^r}$.

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EXISTENCE OF PERIODIC SOLUTIONS FOR A SINGLE-SPECIES POPULATION MODEL WITH ITERATIVE TERMS

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ABSTRACT. The key object of this work lies in establishing some criteria that guarantee the existence of positive periodic solutions for a single-species population model with iterative terms. We use the Green's method and the Krasnoselskii's fixed point theorem for a sum of two mappings. The obtained findings are new and complement some known studies.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 34K30, 34L30.

KEYWORDS AND PHRASES. Population model, Periodic solution, Krasnoselskii's fixed point theorem.

1. PINPOINTING THE PROBLEM

In this paper, we consider the following single-species population model with iterative terms:

$$\frac{d}{dt}x(t) = -b_1(t)\,x(t) + b_2(t)\,x^{[2]}(t) + \frac{d}{dt}f\left(t, x(t), x^{[2]}(t)\right),$$

where $a, b : \mathbb{R} \longrightarrow \mathbb{R}_+^*$, are periodic continuous functions and $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a periodic continuous function. This model can describe the growth of a single species population where the state variable $x(t)$ represents the total number of individuals present at time t, $b_1(t)$ is the death rate, $b_2(t)$ is the birth rate and the second iterate $x^{[2]}(t) = (x \circ x)(t)$ emerges from a time and state dependent delay $\tau(t, x(t))$ differing from one population to another (gestation period, the maturation period, developmental cycle, etc.).

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ELASTIC MEMBRANE EQUATION WITH DYNAMIC BOUNDARY CONDITIONS AND INFINITE MEMORY

MERAH AHLEM[1] AND MESLOUB FATIHA[2]

Abstract. This work is concerned with the following viscoelastic wave equation with dynamic boundary conditions,source term and a nonlinear weak damping localized on a part of the boundary and past history:

$$u_{tt}(t) - \left(a + b\left\|\nabla u\right\|^2 + \sigma\left(\nabla u, \nabla u_t\right)\right)\Delta u(t) - \int_0^\infty g(s)\Delta u(t-s)ds = 0,$$

together with boundary conditions

$$\begin{aligned} u(t) &= 0, \quad on\ \Gamma_0 \times \mathbb{R}^+, \\ u_{tt}(t) &= -\frac{\partial u(t)}{\partial \nu} - \frac{\partial u_t(t)}{\partial \nu} + \int_0^\infty g(s)\frac{\partial u}{\partial \nu}(t-s)ds - h(u_t) - f(u), \quad on\ \Gamma_1 \times \mathbb{R}^+ \end{aligned}$$

and initial conditions

$$u(x,-t) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega$$

where $\Omega \subseteq \mathbb{R}^n\,(n \geq 1)$ is a regular and bounded domain. $\partial\Omega = \Gamma_0 \cup \Gamma_1$,$mes\left(\Gamma_0\right) > 0$, $\Gamma_0 \cap \Gamma_1 = \phi$, and $\frac{\partial}{\partial\nu}$is the unit outer normal derivative.Under some appropriate assumptions on the relaxation function.the general decay for the energy have been established using the perturbed Lyapunov functionals and some properties of convex functions.

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Key words and phrases. Elastic membrane equation, Energy decay, Balakrishnan-Taylor damping, Dynamic boundary conditions, Infinite memory.

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EXISTENCE OF WEAK SOLUTION FOR FRACTIONAL DIFFUSION-CONVECTION-REACTION SYSTEM

SARA DOB, MESSAOUD MAOUNI, AND HAKIM LAKHAL

ABSTRACT. Fractional differential equations involve derivatives of fractional order are important mathematical models of some practical problems in many fields such as image denoising, chemistry physics, heat conduction and many other branches of science. In consequence, the subject of fractional differential equations is gaining much importance and attention. In this work we study the existence of weak solutions for nonlinear fractional system with Direchlet boundary conditions. We use a topological method which is based on the Leray-Schauder degree to obtain the result of the existence of solutions.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 35J60, 35D30.

KEYWORDS AND PHRASES. Nonlinear elliptic equations, fractional divergence, weak solution.

1. DEFINE THE PROBLEM

Many methods have been proposed to deal with nonlinear fractional systems: fixed point method, semigroups method, sub-supersolution method, Brouwer degee and Leray-Schauder degree, etc. The last method is an important topological tool introduced by Leray and Schauder in the study of nonlinear partial differential equations in the early 1930s. The nontriviality of the degree ensures the existence of a fixed point of the compact mapping in the domain. It combines the properties of homotopy invariance and additivity, which make the topological tool more convenient in application.

This work is devoted to the study of the existence of solutions to nonlocal equations involving the fractional divergence, we give an application of the Leray-Schauder degree theorem to prove the existence of a weak solution to the following diffusion-convection-reaction system

$$(\mathcal{P})\begin{cases} (u,v)\in U \\ \int a_1(x,u(x))\nabla^s u(x).\nabla^s\varphi(x)dx \quad +\int_\Omega G_1(x)g_1(u(x)).\nabla^s\varphi(x)dx \\ \qquad = \int_\Omega f_1(x,u(x)).\nabla^s\varphi(x)dx, \\ \\ \int a_2(x,v(x))\nabla^s v(x).\nabla^s\phi(x)dx \quad +\int_\Omega G_2(x)g_2(v(x)).\nabla^s\phi(x)dx \\ \qquad = \int_\Omega f_2(x,v(x)).\nabla^s\phi(x)dx, \\ \text{for all } (\varphi,\phi)\in U \end{cases}$$

1

which is the weak formulation of the following system:

$$(\mathcal{PV})\begin{cases} -div^s(a_1(x,u(x))\nabla^s u(x))) - div^s(G_1(x)g_1(u)) = f_1(x,u(x)) \text{ in } \Omega, \\ \\ -div^s(a_2(x,v(x))\nabla^s v(x)) - div^s(G_2(x)g_2(v)) = f_2(x,v(x)) \text{ in } \Omega, \\ \\ u = v = 0 \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

We place ourselves under the following assumptions:

$$(\mathcal{HP})\begin{cases} (i) & \Omega \text{ is a bounded open of } \mathbb{R}^n, \quad n \geq 1 \text{ and } s \in]0,1[, \\ (ii) & a_i \text{ are Carathodory functions }, \\ (iii) & \exists \alpha_i, \beta_i \in \mathbb{R}; \alpha_i \leq a_i(x,z) \leq \beta_i \quad \forall z \in \mathbb{R} \quad a.e. x \in \Omega, \\ (iv) & G_i \in C^1(\bar{\Omega}, \mathbb{R}^n), \quad div^s G_i = 0, \\ (v) & g_i \in C(\mathbb{R},\mathbb{R}) \text{ and there are } C_i \geq 0 \text{ such as } |C_i(z)| \leq |z| \quad \forall z \in \mathbb{R}, \\ (vi) & f_i \text{ are Carathodory functions }, \text{ and } \exists L_i \geq 0 \text{ and } d_i \in L^2(\Omega); \\ & |f_i(x,z)| \leq d_i(x) + L_i(z), \\ (vii) & \lim\limits_{z \to \pm\infty} \frac{f_i(x,z)}{z} = 0, \end{cases}$$

for all $i = 1, 2$.

The main result of this work is

Theorem 1.1. *Under hypothesis (HP), system (P) has a solution* $(u, v) \in U$.

2. Proof of the main result

In this section, we study the existence of a weak solution for the nonlinear fractional system with Dirichlet boundary conditions. We use the Leray-Schauder degree to solve a diffusion-convection-reaction system.
This method requires a priori estimates, i.e. estimates on (u, v), without knowing its existence.
we will define a homotopy H and we will verify the three condition of the Leray-Schauder degree to arrive at existence result.

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EXISTENCE RESULTS FOR A SEMILINEAR SYSTEM OF DISCRETE EQUATIONS

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Abstract. In this work, we establish several results about the existence and uniqueness of solutions for some classes of semilinear systems of difference equations with initial and boundary conditions. The approach is based on a fixed point theory in vector-valued Banach spaces. Also, we give an abstract formulation to Sadovskii's fixed point theorem in vector-valued Banach space.

2010 Mathematics Subject Classification. 34K45, 34A60

Keywords and phrases. Discrete system, fixed point, generalized metric space, condensing operator.

1. the problem

In this we consider the semilinear discrete system of the form:

$$(1)\qquad \begin{cases} x(t) &= A(t)x(t) + f_1(t, x(t), y(t)),\ k \in \mathbb{N}(a,b),\\ y(t) &= A(t)y(t) + f_2(t, x(t), y(t)),\ k \in \mathbb{N}(a,b),\\ x(a) &= x_0,\\ y(a) &= y_0, \end{cases}$$

where $\mathbb{N}(a,b) = \{a, a+1, \dots, b+1\}$, $f_1, f_2 : \mathbb{N}(a,b) \times X \to X$ are given functions and with a variable linear operator $A(t)$ in a Banach space X.

Later, we study the following impulsive boundary-value problems:

$$(2)\qquad \begin{cases} x(t) &= A(t)x(t) + f_1(t, x(t), y(t)),\ k \in \mathbb{N}(0,b),\\ y(t) &= A(t)y(t) + f_2(t, x(t), y(t)),\ k \in \mathbb{N}(0,b),\\ L_1(x(0)) &= l_1 \in X,\\ L_2(y(0)) &= l_2 \in X, \end{cases}$$

wher $L_1, L_2 : C(\mathbb{N}(0,b), X) \to X$ are two bounded linear operator.

This paper is organized as follows. In Section **??**, we introduce all the background material needed such as generalized metric spaces and some fixed point theorems. In Section **??**, by using the measure of noncompactness, we prove some Sadovskii fixed point theorems. The existence and uniqueness of solutions to the problems (1) and (2) are studied in Sections **??** and **??**, respectively.

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Ergodicity in Stepanov-Orlicz spaces

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Abstract : The aim of this work is to introduce new classes of functions called Stepanov-Orlicz ergodic functions, which generalize in a natural way the classical Stepanov ergodicity introduced by Diagana. Comparative study of these new functions is investigated. Examples and counterexamples are presented.

Key words : Ergodicity, Stepanov-Orlicz spaces, Luxemburg norm ergodicity, Modular ergodicity, Stepanov-Orlicz ergodicity, Lebesgue space with variable exponents

Classification MSC2010 : 46E30, 47A30

In the early nineties, Zhang [9] introduced a significant generalization of almost periodic functions, the so called pseudo almost periodic functions by disturbing the almost periodic function by an ergodic term. Namely, a bounded continuous complex-valued function f is said to be ergodic if it satisfies

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|f(t)|\,d\mu(t)=0, \tag{1}$$

where μ denotes the Lebesgue measure on $\mathbb{R}$. Under the boundedness condition which is of metric nature, Blot et al. [1] gave an elegant characterization of (1) via the following topological property: For any $\varepsilon>0$

$$\lim_{r\to+\infty}\frac{1}{2r}\mu\left(\{t\in[-r,r]\,,|f(t)|\geq\varepsilon\}\right)=0 \tag{2}$$

in a general setting, when μ is any Borel measure on $\mathbb{R}$, satisfying $\mu(\mathbb{R})=+\infty$ and $\mu([a,b])<+\infty$, for all $a,b\in\mathbb{R}$, $(a\leq b)$, and for vector-valued functions f. Zhang's ergodicty has undergone various important generalizations, such as weighted pseudo almost periodicity and μ-pseudo almost periodicity introduced by Blot et al. [1, 2] for which the other previous definitions become just a particular cases. In [4], Diagana and Zitane introduce and study a new class of weighted Stepanov-like pseudo-almost periodic functions with variable exponents, which include Stepanov pseudo almost periodicity [3] as special case.

These notions of pseudo almost periodicity and their generalizations have been successfully researched in abstract differential equations, evolution equations, and integro-differential equations because of there theoretical applications in control theory, mathematical biology, etc. We refer the readers to [1, 5, 6, 7] and the references therein.

The direct impetus of this work comes from Diagana and Zitane's paper [4] where a new notion called Stepanov-like pseudo-almost periodic functions in Lebesgue spaces with variable exponents $L^{p(\cdot)}$ is explored. The authors make extensive use of the Lebesgue spaces with variable exponents $L^{p(\cdot)}$ to investigate some fundamental properties of these functions. The $L^{p(\cdot)}$ spaces have the advantage that they are generated by a particular Musielak-Orlicz function $\phi(t,x)=|x|^{p(t)}$ satisfying the Δ_2-condition.

As can be seen in (1) and (2), the notion of ergodicity is based on a topological property and on a metric property (boundedness). Diagana and Zitane [4] extend Zhang's ergodicity definition to Lebesgue spaces with variable exponents $L^{p(\cdot)}$ context by replacing the absolute value in (1) by the Luxemburg norm, and boundedness property by the one in Luxemburg norm sense. This motivated us to consider the case when $L^{p(\cdot)}$ space is equipped with the topology induced by the modular convergence. It turns out that replacing the absolute value by the Luxemburg norm or by its associated modular gives rise to two identical concepts.

Things become even more complicated if one takes the convergence in a general Orlicz spaces. This allows us to see that ergodicity in Stepanov-Orlicz sense can be forks into many different notions when applied to Orlicz spaces: Luxemburg norm ergodicity, modular ergodicity and strongly modular ergodicity in Stepanov Orlicz sense.

Our main objective is to study the hierarchy of those various notions. A comparative study, examples and counterexamples on the new introduced spaces will be investigated.

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ESTIMATES FOR SEMI LINEAR WAVE MODELS WITH TWO DAMPING TERMS.

MOURAD KAINANE MEZADEK

Abstract. In this work we study the global (in time) existence of small data solutions to the Cauchy problem for the semilinear wave equation with friction, visco-elastic damping and a power nonlinearity. We are interested in the connection between regularity assumptions for the data and the admissible range of exponents p in the power nonlinearity $|u_t|^p$.

2010 Mathematics Subject Classification. [2010] 35L05 35L71

Keywords and phrases. global in time existence, small data solutions, wave equation, visco-elastic damping, frictional damping, power nonlinearity, higher regularity of data, fractional chain rule
.

1. Define the problem

In this work we are interested to study the following Cauchy problem for the semilinear wave equation with two types of damping terms, friction and visco-elastic damping as well, and with power nonlinearity:

$$\begin{aligned} &u_{tt} - \Delta u + u_t - \Delta u_t = |u_t|^p \text{ for } (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ &u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x) \text{ for } x \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where the data φ and ψ are given Cauchy data.
Under certain assumptions for the data and the dimension n. Our main goals are to study the influence of regularity parameters $s_1, s_2 \in \mathbb{R}^+$ and additional regularity parameter $m \in [1,2)$ for the data (φ, ψ), that is,

$$(\varphi, \psi) \in (H^{s_1} \cap L^m) \times (H^{s_2} \cap L^m)$$

on the admissible range of exponents p which allow to prove the global (in time) existence of small data Sobolev solutions or energy solutions with suitable regularity.

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Existence and stability for nonlinear Caputo-Hadamard fractional delay differential equations

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Abstract: We use the modified version of contraction mapping principle to obtain the existence and uniqueness of solutions for nonlinear Caputo-Hadamard fractional delay differential equations. We also use the method of successive approximations to show the stability of the equations.

Keywords: Fractional delay differential equations; Caputo-Hadamard fractional derivatives; fixed point theorems; existence; uniqueness; successive approximations; Ulam-Hyers stability; E_α-Ulam-Hyers stability.

1 Introduction

Fractional differential equations with and without delay arise from a variety of applications including various elds of science and engineering such as applied sciences, physics, chemistry, biology, medicine, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors.

In this work, we concentrate on the existence and uniqueness of solutions and stability results for the nonlinear delay fractional differential equation

$$\mathfrak{D}_1^\alpha x(t) = f(t, x_t), \ t \in [1, b], b > 1, \tag{1}$$

$$x(t) = \psi(t), \ t \in [1 - r, 1], \tag{2}$$

where $f : [1, b] \times C([1 - r, 1], \mathbb{R}^n) \to \mathbb{R}^n$ is a nonlinear continuous function, and $\mathfrak{D}_1^\alpha$ denotes the Caputo-Hadamard derivative of order $m - 1 < \alpha \leq m \in \mathbb{N}$.

Let $\mathbb{R}^n$is an n-dimensional linear vector space over the reals with the norm

$$\|x\| = \left(\sum_{k=1}^{n} x_k^2 \right)^{\frac{1}{2}}, \ x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n.$$

Let $0 \leq r < \infty$ be given real number, $C = C([1-r, 1], \ \mathbb{R}^n)$, $b > 1$ the Banach space of continuous functions from $[1-r, 1]$ into $\mathbb{R}^n$ with the norme

$$\|\phi\|_C = \sup_{1-r \leq \theta \leq 1} \|\phi(\theta)\|.$$

Let us denote by $B = C^m([1-r, b], \mathbb{R}^n)$, $b > 1$ the Banach space of all continuous functions from $[1-r, b]$ into $\mathbb{R}^n$ having m^{th} order derivatives endowed with supremum norm $\|\cdot\|_B$. For any $x \in B$ and any $t \in [1, b]$, we denote by x_t the element of C defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [1-r, 1]$.

To show the existence and uniqueness of solutions, we transform (1)-(2) into an integral equation and then use the modified version of contraction principle. Further, by the successive approximation method, we obtain Ulam-Hyers, Ulam-Hyers-Rassias, and E -Ulam-Hyers stability results of (1).

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EXISTENCE DE SOLUTION POUR UN PROBLÈME DE FLUIDES NON NEWTONIEN

EL HACÈNE OUAZAR

ABSTRACT. Dans ce travail nous étudions l'existence d'une solution faible d'un système non linéaire regissant le mouvement d'un fluide non Newtonien. l'ordre de derivation la plus grande (trois) se trouve dans le terme non linéaire et le terme linéaire regularisant Δu est d'ordre (2). Nous cherchons la solution dans $W^{2,p}, p > n$
La clé de la démonstration de l'existence pour ce système est dans la décomposition du problème, en vu d'appliquer la méthode de point fixe sur un compacte

2010 MATHEMATICS SUBJECT CLASSIFICATION. 76D03, 46E35, 35A15.è

KEYWORDS AND PHRASES. Aqueous solution, incompressible fluids, Sobolev spaces, specials basis, slip boundary conditions, Stationary problem, compact operator

1. DEFINE THE PROBLEM

Notre but dans ce travail est de montrer l'existence d'une solution du systme

$$(\mathcal{E}) \begin{cases} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)(\boldsymbol{u} - \alpha\Delta\boldsymbol{u}) + \nabla\pi = \boldsymbol{f}, & in\ \Omega \\ div\boldsymbol{u} = \boldsymbol{0}, & in\ \Omega \end{cases}$$

Ce systéme régi le mouvement d'une solution aqueuse d'un fluide non newtonien incompressible dont la loi de comportement

$$\mathcal{T} = -pI + \sigma,\ \sigma = 2\nu D + 2\alpha\frac{dD}{dt}$$

Dans amrouche-ouazar [1], Il est démontré l'existence d'une solution appartenant à l'espace $H^2(\Omega)$ mais on ne peut pas déduire la régularité $W^{2,p}$ par le procédé utilisé sur le système de Navier-Stokes du fait que la plus grande dérivée se trouve dans le terme non linéaire,comme signalée plus haut. En s'insprant dr la méthode de décomposition utiliée Dans [2] pour le systéme régissant l'écoulement des fluides appelés fluides de grade 2:

$$(\mathcal{E}) \begin{cases} -\nu\Delta \boldsymbol{u} + rot(\boldsymbol{u} - \alpha\Delta\boldsymbol{u}) \times \boldsymbol{u} + \nabla\pi = \boldsymbol{f}, & in\ \Omega \\ div\boldsymbol{u} = 0, & in\ \Omega \end{cases}$$

Dans le quel, Le terme $rot(\boldsymbol{u} - \alpha\Delta\boldsymbol{u}) \times \boldsymbol{u}$ et la propriété fonctionnelle $rot(\nabla\psi) = 0, \forall\psi \in H^1(\Omega)$, sont bien exploitée dans ce problème pour surmonter l'incovénient de l'existence du terme de pression $\nabla\pi$,dans la decomposition du point fixe, , ce qui n'est pas le cas pour notre principale équation. Dans ce travail nous avons utilisé cette idée de décomposition, mais après avoir pu trouver une formulation point fixe adéquate. En plus, avec les résultats de Novotny [3] , on a études ce problèmes avec les conditions au bord de glissement(non d'adhérence au bord) comme c'est fait dans [2]

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EXISTENCE OF A SOLUTION FOR A CLASS OF HIGHER-ORDER BOUNDARY VALUE PROBLEM

SALAH BENHIOUNA AND AZZEDDINE BELLOUR

Abstract. In this paper, we first establish a generalization of Arzelà-Ascoli theorem in Banach spaces, and then use Schauder's fixed point theorem to prove the existence of a solution for the boundary value problem of higher order. Our results are obtained under rather general assumptions.

2010 Mathematics Subject Classification. 34B15, 45J05, 46E15, 47H10.

Keywords and phrases. Generalization of Arzelà-Ascoli Theorem, Higher-order boundary value problem, Fixed point theorem.

1. Define the problem

In this paper, we consider the following higher-order boundary value problems:

$$\begin{cases} u^{(n)} + f(t, u, u', ..., u^{(n-2)}) = 0, n \geq 2, t \in I = [0,1], \\ u^{(i)}(0) = 0, 0 \leq i \leq n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0. \end{cases} \tag{1}$$

where n is a given positive integer, $\alpha, \gamma > 0$ and $\beta, \delta \geq 0$.
Our main task in this paper consists of giving a generalization of Ascoli-Arzela theorem in the space $C^n(X, E)$ (the space of functions from a compact subset of $\mathbb{R}$ into a Banach space E with continuous nth derivative) in order to prove the compactness criteria and to use Schauder fixed point theorem in the space C^n to prove the existences of a solution for the higher-order boundary value problems (1).

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EXISTENCE OF SOLUTION FOR ELLIPTIC PROBLEM WITH SINGULAR NONLINEARITIES

N. ELHARRAR, J. IGBIDA, AND H. TALIBI

ABSTRACT. In the present paper, we prove the existence and uniqueness of weak solutions to a class of $p(.)$-Laplacian problem with Singular Nonlinearities, the main tool used here is using a regularization and Schauders fixed point method with the theory of variables Sobolev spaces.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 35J92; 35J60; 35D30; 35A02.

KEYWORDS AND PHRASES. $p(\cdot)$-laplacian, quasilinear elliptic problem, nonlinear singular terms, existence, weak solution.

1. DEFINE THE PROBLEM

We consider the following $p(\cdot)$-Laplacian problem with nonlinear singular terms

$$-\Delta_{p(x)} u = \frac{f(x)}{u^\alpha},\ in\ \Omega,$$

where $\alpha \geq 1$ and $p(\cdot)$ is a continuous function defined on $\bar{\Omega}$ with Ω is a bounded regular domain in $\mathbb{R}^N, N \geq p(x) > 1$. f is assumed to be a non negative function belonging to a suitable Lebesgue space $L^m(\Omega)$.

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EXISTENCE OF SOLUTIONS FOR FRACTIONAL INTEGRAL BOUNDARY VALUE PROBLEMS OF FRACTIONAL DIFFERENTIAL EQUATION ON INFINITE INTERVAL

ABDELLATIF GHENDIR AOUN

ABSTRACT. In this subject, we are concerned with the existence of solutions to fractional differential equation subject to Riemann-Liouville fractional integral boundary conditions. By means of a recent fixed point theorem, sufficient conditions are obtained that guarantee the existence of at least one solution. An example of application illustrate the applicability of the theoretical result.

2010 MATHEMATICS SUBJECT CLASSIFICATION. xxxx, xxxx, xxxx.

KEYWORDS AND PHRASES. Boundary value problem, fractional differential equation, infinite interval, fixed point theorem.

1. DEFINE THE PROBLEM

In this subject, we will investigate the boundary value problem

$$(1)\qquad \begin{cases} D_{0^+}^{\alpha}u(t) + f(t, u(t), D_{0^+}^{\alpha-1}u(t)) = 0, & t \in (0, +\infty), \\ u(0) = 0, \quad \lim\limits_{t\to+\infty} D_{0^+}^{\alpha-1}u(t) = \beta I_{0^+}^{\alpha-1}u(\eta), \end{cases}$$

where $1 < \alpha \leqslant 2$, $\eta > 0$ and $\beta > 0$ satisfies $0 < \beta\eta^{2\alpha-2} < \Gamma(2\alpha - 1)$. $D_{0^+}^{\alpha}$ refers to the standard Riemann-Liouville fractional derivative and $I_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional integral.

In the last years, much of progress has been made in the study of boundary value problems involving differential equations.

For instance, A. Guezane-Lakoud, R. Khaldi [2] have studied the following boundary value problem with fractional integral boundary conditions in bounded interval

$$\begin{cases} {}^{c}D_{0^+}^{q}x(t) + f(t, x(t), {}^{c}D_{0^+}^{p}x(t)) = 0, \quad 0 < t < 1, \ 1 < q \leqslant 2, \ 0 < p < 1 \\ x(0) = 0, \quad x'(1) = \alpha I_{0^+}^{p}x(1), \end{cases}$$

where ${}^{c}D^{q}$ denotes the Caputo fractional derivative.

In [3], C. Shen, H. Zhou, and L. Yang have established existence of positive solutions for the boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}u(t) + f(t, u(t), D_{0^+}^{\alpha-1}u(t)) = 0, & t \in (0, +\infty), \\ u(0) = 0, \quad u'(0) = 0, \quad D_{0^+}^{\alpha-1}u(+\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases}$$

where $2 < \alpha \leqslant 3$. Using the Schauder fixed point theorem, they have showed the existence of one solution under suitable growth conditions imposed on the nonlinear term.

X. Su and S. Zhang [4] discussed the existence of unbounded solutions and used Schauder's fixed point theorem to prove existence of solutions for the boundary value problem:

$$\begin{cases} D^{\alpha}_{0^+}u(t) + f(t, u(t), D^{\alpha-1}_{0^+}u(t)) = 0, & t \in (0, +\infty), \\ u(0) = 0, \quad u'(0) = 0, \quad D^{\alpha-1}_{0^+}u(\infty) = u_\infty, & u_\infty \in \mathbb{R}, \end{cases}$$

where $1 < \alpha \leqslant 2$.

C. Yu, J. Wang, and Y. Guo [5] have considered the solvability of the following integral boundary value problem of fractional differential equation:

$$\begin{cases} D^{\alpha}_{0^+}u(t) + f(t, u(t), D^{\alpha-1}_{0^+}u(t)) = 0, & t \in (0, +\infty), \\ u(0) = 0, \quad D^{\alpha-1}_{0^+}u(\infty) = \int_\eta^{+\infty} g(t)u(t)dt, \end{cases}$$

where $1 < \alpha \leqslant 2$, $f \in C([0, +\infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\eta \geqslant 0$, $g \in L^1[0, +\infty)$ and $\int_\eta^{+\infty} g(t)u(t)dt < \Gamma(\alpha)$.

The work presented in this subject is a continuation of previous works and is concerned with a boundary value problem of fractional order set on the half-axis. It is mainly motivated by papers [2], [3], [4], [5]. To overcome the difficulty related to the compactness of the fixed point operator, a special Banach space is introduced. Our results allow the integral condition to depend on the fractional integral $I^{\alpha-1}_{0^+}u$ which leads to additional difficulties.

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Abstract:

This paper is dedicated to investigating the following elliptic equation with Kirchho type involving the p-Laplacian operator.

Using the variational methods and critical points theory, we obtain the existence of non-trivial solution.

EXISTENCE RESULTS FOR ELLIPTIC EQUATIONS INVOLVING TWO CRITICAL SINGULAR NONLINEARITIES AT THE SAME POLE

ATIKA MATALLAH AND SAFIA BENMANSOUR

Abstract. In this work, we use variational methods to prove the existence of positive solutions for an elliptic equation with two critical Hardy Sobolev exponents at the same pole and a certain nonlinear perturbation. Some parameters play a crucial role in our work.

2010 Mathematics Subject Classification. 35J20, 35J50, 35B33

Keywords and phrases.Variational methods, Critical Hardy-Sobolev exponents, Palais-Smale condition, Concentration compactness principle, Multiple critical nonlinearities.

1. Define the problem

In this work, we are concerned with the existence of nontrivial solutions to the following elliptic problem:

$$(\mathcal{P})_{\mu,\alpha,\beta,\lambda}\begin{cases} -\Delta u - \dfrac{\mu}{|x|^2}u = \dfrac{1}{|x|^{\alpha}}|u|^{2^*(\alpha)-2}u + \dfrac{1}{|x|^{\beta}}|u|^{2^*(\beta)-2}u + \lambda |u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\,\Omega, \end{cases}$$

where Ω is an open smooth bounded domain of $\mathbb{R}^N$, $N \geq 3$, $0 \in \Omega$; $0 \leq \alpha, \beta < 2$; $\lambda > 0$; $0 \leq \mu < \overline{\mu} := \left(\frac{N-2}{2}\right)^2$, which is the best Hardy constant; $2^*(s) = \frac{2(N-s)}{N-2}$, for $0 \leq s < 2$ is the critical Hardy-Sobolev exponent and $2 \leq q < 2^* = 2^*(0)$.

The study of this type of problems is motivated by its various applications, for example: in quantum mechanics, chemistry, physics and differential geometry, etc. The mathematical interest lies in the fact that these problems are double critical due to the presence of different Hardy Sobolev exponents defined at the same pole of nonlinearities.

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EXISTENCE RESULTS FOR HIGHER ORDER FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

ADEL LACHOURI AND ABDELOUAHEB ARDJOUNI

Abstract. In this work, we obtain some novel existence and uniqueness results for higher order fractional differential equations subject to integral boundary conditions. Our results are obtained via the fixed point theorems. Example is given which illustrate the effectiveness of the theoretical results.

2010 Mathematics Subject Classification. 34A08, 34A12.

Keywords and phrases. Fractional differential equations, existence, uniqueness, integral boundary conditions, fixed point.

1. Define the problem

Motivated by the mentioned works in [1, 2, 3], in this work, we prove the existence and uniqueness of mild solutions for higher order fractional differential equations with integral boundary conditions

$$\begin{cases} {}^{C}D^{\alpha}x(t) = f(t, x(t)), \ t \in (0, T), \\ x(0) = x'(0) = x''(0) = \ldots = x^{(n-2)}(0) = 0, \\ x(T) = \lambda \int_0^T x(s)\, ds + d, \end{cases} \tag{1}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order α, $n-1 < \alpha \leq n$, $n \geq 2$, $n \in \mathbb{N}$, $\lambda, d \in \mathbb{R}$ and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. To obtain our results, We convert the problem (1) into an equivalent integral equation. Then we construct appropriate mappings and employ the Schauder fixed point theorem to show the existence of a mild solution. We also use the Banach fixed point theorem to show the existence of a unique mild solution.

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Exponential decay for a nonlinear axially moving viscoelastic string under a Boundary Disturbance

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Abstract

The stabilization of a nonlinear axially moving viscoelastic string is the topic of this paper. Next, we are showing Under reasonable conditions on the initial results, by using the prospective well process, certain solutions exist globally. We then demonstrate that the damping provided by the viscoelastic term is sufficient to ensure an exponential decay.
Key words: moving string,Arbitrary decay ,multiplier method, Asymptotic behavior, Stability. Stability.

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EXPONENTIAL STABILIZATION OF A THERMOELASTICITY SYSTEM WITH WENTZELL CONDITIONS

H. KASRI

Abstract. In this work, the uniform stabilization of thermoelasticity system with static Wentzell type is considered, and the uniform energy decay rate for the problem is established using multiplier method.

2010 Mathematics Subject Classification. 93C20, 93D15.

Keywords and phrases. Thermoelasticity, exponential stabilization, Wentzell conditions, boundary feedbacks.

1. Define the problem

This paper is devoted to studying the exponential stabilization of the solutions of the electromagneto-elastic system with Wentzell conditions by linear boundary feedbacks. More precisely, let Ω be a bounded domain of $\mathbb{R}^3$ with a boundary $\Gamma = \partial\Omega$ of class C^3. The model is given by:

$$(1) \qquad \begin{cases} u'' - \mathbf{div}\sigma(u) + \xi\nabla\theta = 0, & \text{in } Q = \Omega\times]0,\infty[, \\ \theta' - \Delta\theta + \beta\mathbf{div}u' = 0, & \text{in Q}, \\ \theta = 0, & \text{in Q}, \end{cases}$$

Our notations in (1) are standard:$u' = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial t^2}$, $u(x,t) \in \mathbb{R}^3$ denote the displacement vector at $x = (x_1, x_2, x_3) \in \Omega$ and t is the time variable and $\theta = \theta(x,t)$ represent the temperature. $\sigma(u) = (\sigma_{ij}(u))_{i,j=1}^3$ is the stress tensor given by $\sigma(u) = 2\alpha\varepsilon(u)+\lambda\mathbf{div}(u)I_3$, where λ and α are the Lamé coefficients, I_3 is the identity matrix of $\mathbb{R}^3$ and $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) = [\varepsilon_{ij}(u)]_{i,j=1}^3$ is a 3×3 symmetric matrix. From now on, a summation convention with respect to repeated indexes will be use. Also, in system (1) the coupling parameters ξ and β are supposed to be positive.

We complement system (1) with initial conditions

$$(2) \qquad u(.,0) = u_0,\ u'(.,0) = u_1,\ \theta(.,0) = \theta_0,\ \text{in } \Omega,$$

and boundary conditions

$$(3) \qquad \begin{cases} \sigma_S(u) - \overline{\mathbf{div}_T\sigma_T^0(u)} + au_T + bu' = 0, & \text{on } \Sigma, \\ \sigma_\nu(u) + \sigma_T^0(u) : \partial_m\nu + au_\nu + bu' = 0, & \text{on } \Sigma, \end{cases}$$

where $a = a(x)$ and $b = b(x)$ be two nonnegative functions belongs to $C^1(\Gamma)$, $\sigma_T^0(u) : \partial_m\nu = tr(\sigma_T^0(u) \cdot \partial_m\nu)$, with

$$(4) \qquad \sigma_T^0(u) = 2\alpha\varepsilon_T^0(u) + \frac{2\lambda\alpha}{\lambda + 2\alpha}tr(\varepsilon_T^0(u))I_2,$$

and "tr" means the trace of a matrix. As usual $\nu = \nu(x)$ denotes the unit normal vector at $x \in \Gamma$ pointing the exterior of Ω.

Wentzell boundary conditions are characterized by the presence of tangential differential operators of the same order as the interior operator. These boundary conditions are usually justified by asymptotic methods and appear in several fields of applications such as physics, in diffusion processes [17], in mechanics [12], as well as in wave phenomena [4]. Such a system was first investigated by Lemrabet [12] and subsequently by A. Heminna [4, 3, 5] and Kasri [9, 10]. In [5] the author showed that the *natural* feedback is not sufficient to guarantee the exponential decay of the energy $(\mathcal{E}(t))$ in the case of the wave equation with Wentzell conditions.

In [14], W. Lui considered the thermoelastic system with the following boundary conditions

$$\mu\frac{\partial u}{\partial \nu} + (\lambda+\mu)\mathbf{div}(u)\nu + a(x)m\cdot\nu u + (m\cdot\nu)u' = 0$$

Under suitable geometric conditions imposed on the domain, he proved results of stabilization and exact controllability for the model. Later on, [15] treated the above problem in the case when the linear boundary feedback term $(m\cdot\nu)u'$ is replaced by the nonlinear velocity boundary damping $(m\cdot\nu)g(u')$, by using the multiplier techniques and suitable Lyapunov functionals, they established both exponential and polynomial decay rates for the energy. This work was later improved by [6], they considered the situation where the localized internal nonlinear velocity feedback acts effective in the whole domain Ω and the nonlinear boundary velocity damping acts on a part of Γ. Their proof is based on the multiplier method combined with nonlinear integral inequalities to show that the energy of the system decays to zero as t goes to infinity.

Quite recently, H. Kasri and A. Heminna [7](resp.[8]) considered a coupled Maxwell/wave system (resp. electromagneto-elastic system) with Wentzell conditions and proved that the energy of the system decays exponentially if Ω is strictly star shaped with respect to the origin. Their method of proof is based on the validity of some stability estimate which is obtained using the multiplier method.

Therefore one may ask, Does $(\mathcal{E}(t))$ the energy of the system $(1)-(3)$ tend to zero exponentially as time goes to infinity?

Our aim in this work is to investigate $(1)-(3)$ and establish exponential decay result, i.e., explicit energy decay rates.

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Fixed point theorems in the study of positive strict set-contractions

Salima Mechrouk

Abstract

The author uses fixed point index properties and Inspired by the work in Benmezai and Boucheneb (see Theorem 3.8 in [3]) to prove new fixed point theorems for strict set-contraction defined on a Banach space and leaving invariant a cone.

Key words: Cones, fixed point theory, strict set-contractions, positive solution, general minorant principle, boundary value problem.
2010 Mathematics Subject Classifications: 47H10, 47H11, 47H30.

1 Introduction

In the study non-linear operators in ordered Banach spaces having an invariant cone it is often convenient to make use of minorants, majorants and the special concept of the derivatives in order to establish the existence of non-zero fixed points. Krasnoselskii has provided in [10] many interesting fixed point theorems stating that if such an operator is approximatively linear at 0 and $+\infty$, and the spectral radii of the linear approximations are oppositely located with respect to 1, then it has a fixed point. Amann in [2] has generalized these results for monotones operators which are strict set-contractions.

The main goal of this paper is to study strict-set contraction in ordered Banach spaces having an invariant cone and to give sufficient conditions on minorants and majorants which yield the existence of at least one non-zero fixed point (see [4], [3], [1] and [5]). We will assume that the mapping T has an asymptotically linear majorant h and has a minorant g which is right differentiable at zero and existence of the fixed point is obtained under additional conditions about the positive spectra of the derivatives. The proofs are based on the fixed point index theory, developed in [12] (see also the monographs [7] and [8]). In order to be more precise, let X be a Banach space, C be a cone in X, and let $T : C \longrightarrow C$ be a completely continuous mapping. Recently, Mechrouk have proved in [11] that if T has a positive right differentiable at zero minorant $h : K \longrightarrow K$ and an asymptotically linear positive majorant $g : P \longrightarrow P$ satisfying $\theta_P^{g'(\infty)} < 1 < \lambda_P^{h'(0)}$, then T has at least one positive nontrivial fixed point, where the constants $\lambda_P^{h'(0)}$ and $\theta_P^{g'(\infty)}$ play an important role in the statement of the obtained existence and nonexistence results and sometimes they replace the positive spectral radius. Motivated by the above work, we consider in this paper the case where the operator T is a strict set-contraction.

The paper is organized as follows. Section 2 gives some preliminaries. Section 3 is devoted to prove new fixed point theorems for positive maps having approximative minorant and majorant at 0 and ∞ in specific classes of operators. Applications to the existence of solutions to a third order boundary value problem with mixed boundary conditions are presented in the last section.

2 Abstract Background

We will use extensively in this work cones and the fixed point index theory, so let us recall some facts related to these two tools. Let X be a real Banach space endowed with norm $\| \, . \, \|$, and let $L(X) = L(X, X)$ be the set of all linear continuous mapping from X into X. A nonempty closed convex subset C of X is said to be a cone if $(tC) \subset C$ for all $t \geq 0$ and $C \cap (-C) = \{0_X\}$. It is well known that a cone C induces a partial order in the Banach space X. We write for all $x, y \in X : x \preceq y$ if $y - x \in C$, $x \prec y$ if $y - x \in C$, $y \neq x$ and $x \npreceq y$ if $y - x \notin C$. Notations $\succeq$, $\succ$ and $\nsucceq$ denote respectively the reverse situations. We say that the cone C is normal with a constant $n_C > 0$ if for all u, v in C, $u \preceq v$ implies $\|u\| \leq n_C \|v\|$.
Let C be a cone in X and let $L : X \to X$.

Definition 2.1 *The mapping L is said to be positive if $L(C) \subset C$. In this case, a non-negative constant μ is said to be a positive eigenvalue of L if there exists $u \in C \setminus \{0_X\}$ such that $Lu = \mu u$.*

Definition 2.2 *Let A be a nonempty set and let B be an ordered set. A map $g : A \longrightarrow B$ is said to be a majorant of the map $f : A \longrightarrow B$ if $f(x) \leq g(x)$ for all $x \in X$. Minorant is defined by reversing the above inequality sign.*

Definition 2.3 *Let C be a cone in X and $L : X \to X$ a continuous map. L is said to be*
a) *positive, if $L(C) \subset C$,*
b) *strongly positive, if C has a nonempty interior ($intC \neq \emptyset$) and $L(C \setminus \{0_X\}) \subset intC$,*
c) *increasing, if for all $u, v \in X$, $u \preceq v$ implies $Lu \preceq Lv$,*
d) *1-homogeneous, if for all $u \in X$ and $t \in \mathbb{R}$, $L(tu) = tL(u)$.*

Definition 2.4 *Let $L_1, L_2 : X \to X$ be positive maps. We write $L_1 \preceq L_2$ if for all $x \in C$, $L_1 x \preceq L_2 x$.*

Definition 2.5 *Let $B(X)$ be the set of all bounded subsets of X and $\psi : B(X) \longrightarrow \mathbb{R}^+$ be a measure of non-compactness on X; that is ψ satisfies for A, $B \in B(X)$*

1. *$\psi(A) = 0 \Longleftrightarrow A$ is relatively compact on X.*
2. *$A \subseteq B$ imply $\psi(A) \leq \psi(B)$.*
3. *$\psi(\bar{co}A) = \psi(\overline{A}) = \psi(A)$.*
4. *$\psi(A \bigcup B) = \max\{\psi(A), \psi(B)\}$.*
5. *for all $t \in [0, 1], \psi(tA + (1-t)B) = t\psi(A) + (1-t)\psi(B)$*
6. *if $(A_n)_n \subset B(X)$ is a decreasing sequence of closed nonempty sets with $\lim \psi(A_n) = 0$, then $\cap_{n \geq 1} A_n$ is a nonempty compact set.*

Definition 2.6 *A function $f : \Omega \subset X \to X$ is said to be a strict-set contraction if it is continuous, bounded, and there exists a constant $k \in [0, 1)$ such that $\psi(f(S)) \leq k\psi(S)$ for all bounded sets $S \subset \Omega$.*

Proposition 2.7 *(Darbo)*
Let (X_1, d_1) and (X_2, d_2) be metric spaces and $f : X_1 \longrightarrow X_2$ a continuous map.
a) if f is a k-contraction, then f is a k-set contraction,
b) If f is compact on bounded sets, then f is a 0-set-contraction. Conversely, if X_2 is complete and f is a 0-set-contraction, then f is compact on bounded sets.

Definition 2.8 ([13]) *A map $g : C \longrightarrow X$ is said to be differentiable at $x_0 \in C$ along C if there exists $g'(x_0) \in L(X)$ such that*

$$\lim_{h \in C, h \longrightarrow 0} \frac{\| g(x_0 + h) - g(x_0) - g'(x_0)h \|}{\| h \|} = 0.$$

We say that $g'(x_0)$ is the derivative of g at x_0 along C, is uniquely determined.

The map g is said to be asymptotically linear along C if there exists $g'(\infty) \in L(X)$ such that

$$\lim_{x \in C, \|x\| \longrightarrow +\infty} \frac{\| g(x) - g'(\infty)x \|}{\| x \|} = 0.$$

Again, $g'(\infty)$ is uniquely determined and called the derivative at infinity along C.

Lemma 2.9 ([10]) *The derivative $g'(\nu)$, ($\nu = +\infty$, or $x_0 \in C$), with respect to a cone of the positive operator g is a linear positive operator.*

3 Fixed point index

We will make extensive use of fixed point index theory. For the sake of completeness, we recall some basic facts related to this; see [6] and [9].

Let K be a nonempty closed subset of a Banach space X. Then K is called a retract of X if there exists a continuous mapping $r : X \longrightarrow K$ such that $r(x) = x$ for all $x \in K$. Such a mapping is called a retraction. From a theorem by Dugundji, every nonempty closed convex subset of X is a retract of E. In particular, every cone in E is a retract of X. Let K be a retract of X and U be a bounded open subset of K such that $U \subset B(0, R)$, where $B(0, R)$ is the ball centered at 0 of radius R. For any completely continuous mapping $f : \overline{U} \longrightarrow K$ with $f(x) \neq x$ for all $x \in \partial U$, the integer given by

$$i\left(f, U, K\right) = deg\left(I - f \circ r, B(0, R) \cap r^{-1}(U), 0\right).$$

where deg is the Leray-Schauder degree, is well defined and is called the fixed point index. The fixed point index satisfies:

- Normalisation: $i\left(f,U,K\right)=1$ whenever f is constant on $\bar{U}$.

- Additivity: For any pair of disjoint open subsets U_1, U_2 in U such that f has no fixed point on $\bar{U}\setminus U_1\cup U_2$, we have:

$$i\left(f,U,K\right)=i\left(f,U_1,K\right)+i\left(f,U_2,K\right).$$

- Homotopy invariance: The index $i\left(h(x,t),U,K\right)$ does not depend on the parameter t, $0\leq t\leq 1$ where $h:\partial U\times[0,1]\longrightarrow X$ is a compact mapping and $h(x,t)\neq x$ for every $x\in\partial U$ and $0\leq t\leq 1$.

- Permanance: If Y is retract of X and $f(\bar{U})\subset Y$, then

$$i\left(f,U,K\right)=i\left(f,U\cap Y,Y\right).$$

- Excision property. Let $V\subset U$ an open subset such that f has no fixed point in $\bar{U}\setminus V$, then:

$$i\left(f,U,K\right)=i\left(f,V,K\right).$$

- Existence property. If $i\left(f,U,K\right)\neq 0$, then f has a fixed point in U.

The previous results can be applied in a neat way to give a fixed point index for local strict-set-contractions [12].
Let G be an open subset of a space X and assume $f:G\longrightarrow X$ is a local strict-set-contraction such that $S=\{x\in G,\ f(x)=x\}$ is compact. Using Lemma 1 on [12], there exists an open neighborhood V of S such that $K_\infty(f,V)$ is compact. By the results of the previous section there is defined a generalized fixed point index

$$i(f,G,X)=i(f,V\cap X\cap K_\infty(f,V),X\cap K_\infty(f,V)),$$

where

$$\begin{array}{rcl} K_\infty(f,V) &=& \cap_{n\geq 1}K_n(f,V),\\ K_n(f,V) &=& \overline{co\,g(V\cap K_{n-1}(f,V))},\ n>1,\\ K_1(f,V) &=& \overline{co\,f(V)}. \end{array}$$

denotes the closure of a set and $\bar{co}$ is the convex closure of a set.
The fixed point index satisfies:

- The additive property. Let G be an open open subset of a space X and $f:\bar{G}\longrightarrow X$ a local strict-set-contraction such that $S=\{x\in G,\ f(x)=x\}$ is compact. Assume that $S\subset G_1\cup G_2$ where G_1 and G_2 are two disjoint open sets included in G. Then

$$i(f,G,X)=i_X(f,G_1,X)+i(f,G_2,X).$$

- The homotopy property. Let $I = [0,1]$ and let Ω be an open subset of $X \times I$, $X \in F$. Let $F : \Omega \longrightarrow X$ be a continuous map and assume that F is a local strict-set-contraction in the following sense: given $(x,t) \in \Omega$, there exists an open neighborhood of $(x,t) \in \Omega$, $N_{x,t}$, such that for any subset A of X,

$$\gamma\left(F(N_{x,t} \cap (A \times I))\right) \leq k_{x,t}\,\gamma\left(A\right),\, k_{x,t} < 1.$$

Assume that $S = \{(x,t) \in \Omega : F(x,t) = x\}$ is compact. Then $i_X(F_t, \Omega_t)$ is defined for $t \in I$ and

$$i(F_0, \Omega_0, X) = i(F_1, \Omega_1, X).$$

Detailed presentation of the differentiability with respect to a cone can be found in [10] and [13].

Let us recall some lemmas providing fixed point index computations. Let C be a cone in X. Let for $R > 0$, $C_R = C \cap B\left(0_X, R\right)$ where $B\left(0_X, R\right)$ is the open ball of radius R centred at 0_X, and let ∂C_R be its boundary and consider a strict-set contraction mapping, $f : \overline{C_R} \to C$.

Lemma 3.1 ([7]) *If $fx \neq \lambda x$ for all $x \in \partial C_R$ and $\lambda \geq 1$ then $i\left(f, C_R, C\right) = 1$.*

Lemma 3.2 ([7]) *If there exists $e \succ 0_X$ such that $x \neq fx + te$ for all $t \geq 0$ and all $u \in \partial C_R$ then $i\left(f, C_R, C\right) = 0$.*

From the two Lemma above, we conclude the following Lemma.

Lemma 3.3 *If $fx \not\succeq x$ for all $x \in \partial C_R$ then $i\left(f, C_R, C\right) = 1$.*

Lemma 3.4 *If $fx \not\preceq x$ for all $x \in \partial C_R$ then $i\left(f, C_R, C\right) = 0$.*

A detailed presentation of the fixed point index theory for strict-set contraction mappings can be found in [12].

In all this section E is a real Banach space, K is a nontrivial cone in E and $L(E)$ denote the set of all linear continuous self mapping on E endowed with the norm, $\|L\| = \sup\limits_{\|u\|=1} \|Lu\|$. Let $C^+(E)$ denote the subset of $L(E)$ consisting of all strict set-contraction positive operators. Hereafter $\preceq$ is the order induced by the cone K on E and we set,

$$L_K(E) = \{L \in L(E),\ L \text{ is increasing }\}$$

and

$$C_K(E) = \{L \in L_K(E) : L \text{ is a strict-set contraction}\}.$$

Now, for $L \in L_K(E)$ we define the subset

$$\Theta_P^L = \{\theta \geq 0 : \text{ there exists } u \in P \setminus \{0_E\} \text{ such that } Lu \succeq \theta u\}.$$

Remark 3.5 *Note that*

i) $0 \in \Theta_P^L$ *and if* $\theta \in \Theta_P^L$, *then* $[0, \theta] \subset \Theta_P^L$.

ii) $\Lambda_P^L \subset \Lambda_K^L$ *and* $\Theta_P^L \subset \Theta_K^L$.

iii) If μ *is positive eigenvalue of* L, *then* $\mu \in \Theta_P^L \cap [0, \| L \|]$.

iv) If $L^{-1}(0_E) \cap K = \{0_E\}$ *and* $P \subset K$ *then* $\Theta_P^L = \Theta_K^L$.

In all this paper, we set for $L \in L_K^P(E)$,

$$\theta_P^L = \sup \Theta_P^L$$

The constant θ_P^L replaces the spectral radius of L which in our case is not necessarily an eigenvalue of L having an eigenvector in K. So, it is natural to ask what represents this constant with respect to the operator L.
If $L : E \longrightarrow E$ is a bounded linear operator, then we define, $r(L)$, its spectral radius by

$$r(L) = \lim_{n \longrightarrow +\infty} \| L^n \|^{\frac{1}{n}}.$$

Lemma 3.6 gives sufficient conditions for the existence of θ_P^L.

Lemma 3.6 ([3]) *Assume* $L \in L_K(E)$. *Then the subset* Θ_P^L *is bounded from above by* $r(L)$.

Lemma 3.7 ([3]) *Assume that the cone* K *is solid, and let* $L \in C_K(E)$ *be strongly positive and increasing. Then* θ_K^L *is the unique positive eigenvalue of* L.

4 Main results

Theorem 4.1 *Suppose that* T *has a right differentiable at zero majorant* $g : K \longrightarrow K$ *such that* $g(0) = 0$, $g'(0) \in C_K(E)$ *satisfying* $r(g'(0)) < 1$ *and* K *is a normal cone. Then* T *has at least one positive nontrivial fixed point.*

Arguing as in the proof of Theorem 4.1, we obtain the following result.

Theorem 4.2 *Suppose that* T *has an asymptotically linear majorant* $g : K \longrightarrow K$ *such that* $g'(\infty) \in C_K(E)$ *satisfying* $r(g'(\infty)) < 1$ *and* K *is normal. Then* T *has at least one positive nontrivial fixed point.*

Theorem 4.3 *Suppose that the cone* K *is a normal cone and* T *has an asymptotically linear majorant* $g : K \longrightarrow K$ *such that* $g'(\infty) \in C_K(E)$. *Suppose that* T *has a right differentiable at zero minorant* $h : K \longrightarrow K$ *such that* $h(0) = 0$ *and* $h'(0) \in C_K(E)$ *satisfying* $r(g'(\infty)) < 1 < \theta_P^{h'(0)}$. *Then* T *has at least one positive nontrivial fixed point.*

4.1 Application to third order bvp

The aim of this section is to study existence of positive solutions for the following third order boundary value problem

$$\begin{cases} -u'''(t) + c\,u'(t) = a(t)\, f(t, u(t)) & 0 < t < 1 \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{4.1}$$

where c is a positive constant.
Suppose that
(H_1) $a \in C([0,1], \mathbb{R}^+)$ does not vanish identically on any subinterval of $[0,1]$.
(H_2) $f \in C[[0,1] \times \mathbb{R}^+, \mathbb{R}^+]$, $f(t,0) = 0$, $\forall t \in [0,1]$ and for any $l > 0$, $f(t,x)$ is uniformly continuous and bounded on $[0,1] \times (\mathbb{R}^+ \cap S_l)$ and there exists a constant L_l with $0 \le L_l < \frac{c}{2}$ such that

$$\psi(f(t,S)) \le L_l \psi(S),\ \forall t \in [0,1],\ S \subset P \cap S_l,$$

where $S_l = \{x \in \mathbb{R}, \mid x \mid < l\}$ and here ψ denotes the Kuratowski measure of non-compactness on S.
We also consider the associated linear eigenvalue problem

$$\begin{cases} -u'''(t) + c\,u'(t) = \mu\, a(t)\, u(t) & 0 < t < 1 \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{4.2}$$

The Green's function associated with (4.1) given by:

$$G(t,s) = \frac{1}{c \sinh(\sqrt{c})} \begin{cases} \left[\cosh(\sqrt{c}\,t) - 1\right] \sinh(\sqrt{c} - \sqrt{c}\,s) & t \le s, \\ \sinh(\sqrt{c}) - \sinh(\sqrt{c} - \sqrt{c}\,s) \\ \quad - \sinh(\sqrt{c}\,s) \cosh(\sqrt{c} - \sqrt{c}\,t). & s \le t \end{cases} \tag{4.3}$$

We may prove the following Lemma.

Lemma 4.4 *The Green's function $G(t,s)$ possesses the following properties:*

1 $G(.,s)$ and $G(t,.)$ are continuous on $[0,1]$ and

$$G(t,s) \le \frac{1}{c},\ \forall t,s \in [0,1].$$

2 For $s \in (0,\ 1)$ fixed we have

$$\frac{\partial G(t,s)}{\partial t} > 0,\ \forall t \in (0,1),\ \frac{\partial G(0,s)}{\partial t} = 0 \quad \text{and} \quad \frac{\partial G(1,\ s)}{\partial t} = 0.$$

Furthermore $\dfrac{\partial G(t,0)}{\partial t} = \dfrac{\partial G(t,1)}{\partial t} = 0$ *for all* $[0,\ 1]$.

3 For all $t,\ s \in [0,\ 1]$, *we have*

$$\frac{\partial^2 G(t,s)}{\partial t^2} > 0 \ \text{if} \ t \le s \quad \text{and} \quad \frac{\partial^2 G(t,s)}{\partial t^2} < 0 \ \text{if} \ \ s \le t.$$

4 Let $s \in [0,\ 1]$ *fixed. We have*

$$\max_{t\in[0,\ 1]} \mid \frac{\partial^2 G}{\partial t^2}(t,s) \mid = \max\left(\frac{\partial^2 G}{\partial t^2}(0,s),\ -\frac{\partial^2 G}{\partial t^2}(1,s) \right).$$

In all this section, we let E be the Banach space of all continuous functions defined on $[0,1]$ equipped with its sup-norm (for $u \in E$, $\|u\| = \sup\{|u(t)| : t \in [0,1]\}$) and K be the normal cone of nonnegative functions in E.

Lemma 4.5 *The linear eigenvalue problem* (4.2) *has a unique positive eigenvalue* $\mu_\star > 0$.

Let introduce the following notations

$$f^0 = \limsup_{u\to 0} \left(\max_{0\le t\le 1} \frac{f(t,u)}{u} \right) \quad f^\infty = \limsup_{u\to \infty} \left(\max_{0\le t\le 1} \frac{f(t,u)}{u} \right)$$

Theorem 4.6 *The problem* (4.1) *admits a positive solution whenever one of the following conditions:*

$$f^\infty < \mu_\star < f^0 \tag{4.4}$$

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FUNDAMENTAL PROPERTIES RELATED TO CERTAIN OPERATORS ON HILBERT SPACES

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Abstract. The aim of the present talk is to generalize certain properties of parahyponormal operators showed by authors in [4] to a large class of (M,k)-quasi-parahyponormal operators where we present their matrix representation, their finite ascent and and their SVEP.

2010 Mathematics Subject Classification. 47A30, 47B47, 47B20.

Keywords and phrases. Parahyponormal operators, (M,k)-quasi-parahyponormal operators, Ascent and descent of an operator, Single Valued Extension Property.

1. Definition of the problem

In [4], is introduced a class of parahyponormal operators, i.e., operators satisfying $(\mathcal{A}\mathcal{A}^\star)^2-2\lambda\mathcal{A}^\star\mathcal{A}+\lambda^2 \geq 0$ for all $\lambda > 0$, where $\mathcal{A}$ is a bounded linear operator on a complex separable Hilbert space, and are proved some related results. In this talk, we give an extension of these properties to a large class of (M,k)-quasi-parahyponormal operators. The matrix representation, the ascent, the SVEP and other related results are shown.

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GENERAL DECAY OF SOLUTIONS IN ONE-DIMENSIONAL POROUS-ELASTIC SYSTEM WITH MEMORY AND DISTRIBUTED DELAY TERM WITH SECOND SOUND

FARES YAZID, DJAMEL OUCHENANE, AND FATIMA SIHAM DJERADI

Abstract. We investigate a one-dimensional porous-elastic system with the presence of both memory and distributed delay terms in the second equation with second sound. Using the well known energy method combined with Lyapunov functionals approach, we obtain a general decay result.

2010 Mathematics Subject Classification. 35B40, 35L70, 93D15, 93D20.

Keywords and phrases. Porous system, General decay, Exponential Decay, Memory term, Distributed delay term.

1. Define the problem

In this work, we are consider the following problem

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g(t-s)\phi_{xx}(x,s)ds + \mu_1\phi_t \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, \phi_t(x,t-s)ds + \gamma\theta_x = 0, \\ \rho_3\theta_t + \kappa q_x + \gamma\phi_{tx} = 0, \\ \tau_0 q_t + \delta q + \kappa\theta_x = 0. \end{cases}$$

Where $(x,s,t) \in H$, when $H = (0,1) \times (\tau_1,\tau_2) \times (0,\infty)$

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GLOBAL EXISTENCE AND UNIQUENESS OF THE WEAK SOLUTION IN THIXOTROPIC MODEL

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ABSTRACT. In this paper, we study global existence, uniqueness and boundedness of the weak solution for the system (P) which is formulated by two subsystems (P_1) and (P_2), the first describes the thixotropic problem and the second describes the diffusion degradation of c, using Galerkin's method, Lax-Milgran's and maximum principle. Moreover we show that the unique solution is positive.

2010 MATHEMATICS SUBJECT CLASSIFICATION. Primary 90C57, 90C59 Secondary 90C49.

KEYWORDS AND PHRASES. global solution, boundedness, positive solution.

1. DEFINE THE PROBLEM

Our model is defined as follows:

$$(P)\begin{cases} (P_1)\begin{cases} u_t + \Delta u - \lambda div\left[u\frac{\nabla(c-u_0)}{\sqrt{\beta+|\nabla(c-u_0)|^2}}\right] + u = u_0 & (t,x)\in\mathbb{R}^+\times\Omega \\ u = 0 & \partial\Omega \\ u(0,x) = u_0 & x\in\Omega \end{cases} \\ (P_2)\begin{cases} -\Delta c + \tau c = 0 & x\in\Omega \\ c = g & \partial\Omega \end{cases} \end{cases}$$

Where $u(t,x)$ is a function denotes the speed of fluid in the position $x \in \Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$, Ω is a bounded convex domain with smooth boundary $\partial\Omega \in H^{\frac{3}{2}}(\partial\Omega)$, $\lambda > 0$ is the viscosity of the fluid, $\beta > 0$ is a parameter constant, c denotes the concentration of chemical signal that stimulates the fluid. The parameter τ is a time constant and it is expressed on the one hand the movement of fluid and secondly the diffusion degradation of c.

To simplify the solution of the system (P), a decomposition of (P) into two subsystem (P_1) and (P_2) are adopted. Galerkin's method is very important to help us to demonstrate the existence and uniqueness of a weak solution for system (P_1). To prove the existence and uniqueness of a weak solution for system (P_2), we use Lax-Milgram's theorem and maximum principle. However this theorem can not be applied directly because it is nonhomogenous system. For this reason an adoptation of Trace theorem it used to simplify the system(P_2). Therefore we have the existence and uniqueness of a weak solution for system (P). Moreover we show that the solution is positive.

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GENERALIZED WEAKLY SINGULAR INTEGRAL INEQUALITIES WITH APPLICATIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

SALAH BOULARES AND BOUCENNA DJALAL

Abstract. In this research work, we have established some new weakly singular integral inequalities, our obtained inequalities generalize some recent obtained inequalities in the literature. Also, involving a Caputo type fractional derivative with respect to another function, we have obtained some applications to fractional differential equations.

Keywords and phrases. integral inequalities, fractional differential equations.

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GLOBAL EXISTENCE RESULTS FOR SECOND ORDER NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATION WITH STATE-DEPENDENT DELAY

MOUFFAK BENCHOHRA AND IMENE MEDJADJ

ABSTRACT. Our aim in this work is to provide sufficient conditions for the existence of global solutions of second order neutral functional differential equation with state-dependent delay. We use the semigroup theory and Schauder's fixed-point theorem.

2010 MATHEMATICS SUBJECT CLASSIFICATION.34G20, 34K20, 34K30.

KEYWORDS AND PHRASES. Neutral functional differential equation of second order, mild solution, infinite delay, state-dependent delay fixed point, semigroup theory, cosine function.

1. DEFINE THE PROBLEM

we will consider the following problem

$$\frac{d}{dt}[y'(t) - g(t, y_{\rho(t,y_t)})] = Ay(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J := [0, +\infty) \tag{1}$$

$$y(t) = \phi(t),\ t \in (-\infty, 0],\quad y'(0) = \varphi, \tag{2}$$

where $f, g : J \times \rfloor\dashv\Updownarrow\mathcal{B} \to E$ is given function, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t\in\mathbb{R}}$, on E, $\phi \in \rfloor\dashv\Updownarrow\mathcal{B}$, $\rho : J \times \rfloor\dashv\Updownarrow\mathcal{B} \to (-\infty, +\infty)$, and $(E, |.|)$ is a real Banach space. We denote by y_t the element of $\rfloor\dashv\Updownarrow\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta), \theta \in (-\infty, 0]$. We assume that the histories y_t belongs to some abstract phases $\rfloor\dashv\Updownarrow\mathcal{B}$.

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GLOBAL UNIQUENESS RESULTS FOR FRACTIONAL PARTIAL HYPERBOLIC DIFFERENTIAL EQUATIONS WITH INFINITE STATE-DEPENDENT DELAY

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ABSTRACT. In this paper we investigate the existence and uniqueness of solutions of hyperbolic fractional order differential equations with infinite state-dependent delay by using a nonlinear alternative of Leray-Schauder due to Frigon and Granas for contraction maps in Fréchet spaces.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 26A33, 34K30, 34K37.

KEYWORDS AND PHRASES. Partial functional differential equation, fractional order, infinite state-dependent delay.

1. Define the problem

In this work we present a global existence and uniqueness of solutions to the fractional order initial value problem (IVP for short)

$$({}^cD_0^r u)(x,y) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))}),\ \text{if } (x,y)\in J, \tag{1}$$

$$u(x,y) = \phi(x,y),\ \text{if } (x,y)\in \tilde{J}', \tag{2}$$

$$u(x,0) = \varphi(x),\ u(0,y) = \psi(y),\ (x,y)\in J, \tag{3}$$

where $\varphi,\psi : [0,\infty)\to\mathbb{R}^n$, are given absolutely continuous functions, $\varphi(0) = \psi(0)$, $\tilde{J}' =: (-\infty,+\infty)\times(-\infty,+\infty)\backslash[0,\infty)\times[0,\infty)$, $f : J\times\mathcal{B}\to\mathbb{R}$, $\rho_1 : J\times\mathcal{B}\to\mathbb{R}$, $\rho_2 : J\times\mathcal{B}\to\mathbb{R}$ are given functions, $\phi : \tilde{J}'\to\mathbb{R}^n$ is a given continuous function with $\phi(t,0) = \varphi(t)$, $\phi(0,x) = \psi(x)$ for each $(t,x)\in J$ and $\mathcal{B}$ is called a phase space.

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GLOBAL EXISTENCE OF SOLUTION OF NONLINEAR WAVE EQUATION WITH GENERAL SOURCE AND DAMPING TERMS.

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ABSTRACT. In this work, we consider the nonlinear wave equation with general source and damping terms. Using the idea of Salim Messaoudi in (Blow up and global existence in a nonlinear viscoelastic wave equation. Math. Nachr, 260, 58-66, 2003), we prove that the solution is global.

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KEYWORDS AND PHRASES. Wave equation, Damping term, Source term, Global existence.

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HOMOGENIZATION OF THE STOKES PROBLEM

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Abstract. We consider the Stokes problem in a perforated domain in $\mathbb{R}^{\mathbb{N}}, N \geq 3$, with small holes ε-periodically distributed. The size of the holes is of the order $(\varepsilon\delta(\varepsilon))$ with $\delta(\varepsilon) \longrightarrow 0$ as ε goes to zero. On the boundary of the holes we prescribe a Robin-type condition depending on a parameter γ. The aim is to give the asymptotic behavior of the velocity and of the pressure of the fluid as ε goes to zero.
In this work we use the periodic unfolding method introduced by Cioranescu, Damlamian and Griso in [1] and [2] which allows to consider a general geometric framework.
We give the limit problems corresponding to different values of γ (Darcy, Brinkmann or Stokes type problems).

76M50; 34M40; 76S05.

homogenization; periodic unfolding; small holes; Stokes system..

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HIGHER ORDER BOUNDARY VALUED PROBLEM FOR IMPULSIVE DIFFERENTIAL INCLUSIONS

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Abstract. In this paper, we present some existence results for the higher order impulsive differential inclusion:

$$\begin{cases} x^{(n)} \in F(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \ a.e.\ t \in J = [0, \infty)\backslash\{t_1, \dots\}, \\ \Delta x^{(i)}|_{t=t_k} = I_{ik}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)), \ i = 0, 1, \dots, n-1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 1, \dots, \\ x^{(i)}(0) = x_{0i}, \ (i = 0, 1, \dots, n-2), \ x^{(n-1)}(\infty) = \beta x^{(n-1)}(0), \end{cases}$$

where $F : R_+ \times E \times E \times \cdots \times E \to \mathcal{P}(E)$ is a multifunction, $x_{0i} \in E, i = 0, 1, \dots, n-1,\ \ 0 = t_0 < t_1 < \cdots < t_m < \cdots, \lim_{k\to\infty} t_k = \infty,\ I_{ki} \in C(E \times \cdots \times E, E)$ $(i = 1, \dots, n-1,\ k = 1, \dots,)$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-), x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0^+} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively, $x^{(n-1)}(\infty) = \lim_{t\to\infty} x^{(n-1)}(t)$, and $(E, |\cdot|)$ is real separable Banach space.
We present some existence results when the right-hand side multi-valued nonlinearity can be either convex or nonconvex.

2010 Mathematics Subject Classification. 34K45, 34A60, 47D62, 35R12

Keywords and phrases. Impulsive differential inclusions, multi-valued maps,decomposable set, fixed point.

1. Define the problem

Consider nth order impulsive differential inclusions of the form,

(1) $$x^{(n)} \in F(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \quad \text{a.e. } t \in J = [0, \infty)\backslash\{t_1, \dots\}$$

(2)
$$\Delta x^{(i)}|_{t=t_k} = I_{ik}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)), \ i = 0, 1, \dots, n-1, \ k = 1, \dots,$$

(3) $$x^{(i)}(0) = x_{0i}, \ (i = 0, 1, \dots, n-2,), \ x^{(n-1)}(\infty) = \beta x^{(n-1)}(0),$$

where $F : R_+ \times E \times E \times \cdots \times E \to \mathcal{P}E)$ is a multifunction, $x_{0i} \in E, i = 0, 1, \dots, n-1,\ \ 0 = t_0 < t_1 < \cdots < t_m < \cdots, \lim_{k\to\infty} t_k = \infty,\ I_{ik} \in C(E \times \cdots \times E, E)$ $(i = 1, \dots, n-1,\ k = 1, \dots,)$, $\Delta x^{(i)}|_{t=t_k} = x^{(i)}(t_k^+) - x^{(i)}(t_k^-)$, where $x^{(i)}(t_k^+) = \lim_{h\to 0^+} x^{(i)}(t_k+h)$ and $x^{(i)}(t_k^-) = \lim_{h\to 0^+} x^{(i)}(t_k - h)$ represent the right and left limits of $x^{(i)}(t)$ at $t = t_k$, respectively, $x^{(n-1)}(\infty) = \lim_{t\to\infty} x^{(n-1)}(t)$, and $(E, |\cdot|)$ is real separable Banach space.

Our goal in this work is to give some existence results when the right-hand side multi-valued nonlinearity can be either convex or nonconvex. We give an existence result based on nonlinear alternative of Leray-Schauder type for condensing maps (in the convex case), then some existence results are obtained based on the nonlinear alternative of Leray-Schauder type and on the Covitz and Nadler fixed point theorem for contractive multi-valued maps (in the nonconvex case).

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ITERATES OF DIFFERENTIAL OPERATORS OF SHUBIN TYPE IN ANISOTROPIC ROUMIEU GELFAND-SHILOV SPACES

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Abstract. The purpose of this work is to show the iterate property for globally elliptic differential operators with polynomial coefficients (called Shubin operators), in the anisotropic Roumieu Gelfand-Shilov spaces $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$.

2010 Mathematics Subject Classification. Primary 35B65, 35J30, secondary 35H10, 46E10.

Keywords and phrases. Globally elliptic operators, Iterates of operators, Anisotropic Roumieu Gelfand-Shilov spaces, Operators of Shubin type.

1. Define the problem

The aim of this paper is to give an extension of the known iterate theorem of Kotake-Narasimhan [4] in anisotropic Roumieu Gelfand-Shilov classes $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$, for globally elliptic operators with polynomial coefficients, called operators of Shubin type. The iterate problem consists to characterize functional spaces in help with iterates of differential operators, it gives also results of regularity of solutions of partial differential equations in these spaces. At first time the functional spaces considered are those having local properties as different types of Gevrey spaces (see [1, 2, 5]), and Roumieu spaces defined by sequences of positive real numbers (see [3]).

For the definition of Roumieu Gelfand-Shilov spaces $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$, we will consider sequences of positive real numbers (M_p) satisfying the following conditions:

Logarithmic convexity:

$$M_0 = 1,\ M_p^2 \leq M_{p+1}M_{p-1},\ \forall p \in \mathbb{N}^*; \tag{1}$$

Stability under derivation and multiplication:

$$\exists H > 0 : \binom{p+q}{p} M_p M_q \leq M_{p+q} \leq H^{p+q} M_p M_q,\ \forall p, q \in \mathbb{N}; \tag{2}$$

Example 1.1. *The sequence $M_p = p!^s$, $s \geq 1$, satisfies the conditions $(1)-(2)$. It is called Gevrey sequence of order s.*

Definition 1.2. *Let (M_p) and (N_p) two sequences satisfying the conditions $(1)-(2)$, we call anisotropic Roumieu Gelfand-Shilov space and we denote $S_{\{N\}}^{\{M\}}(\mathbb{R}^n)$, the space of all functions $u \in C^\infty(\mathbb{R}^n)$ such that*

$$\exists C > 0,\ \sup_{x\in\mathbb{R}^n} \left| x^\beta \partial^\alpha u(x) \right| \leq C^{|\alpha|+|\beta|+1} M_{|\alpha|} N_{|\beta|},\ \forall \alpha \in \mathbb{N}^n, \forall \beta \in \mathbb{N}^n \tag{3}$$

Let P be a partial differential operator with polynomial coefficients, i.e. of the form

$$P(x,D)=\sum_{|\alpha|+|\beta|\leq m} C_{\alpha\beta}x^{\beta}D^{\alpha},\ C_{\alpha\beta}\in\mathbb{C},\ \text{and } D^{\alpha}=(i)^{-|\alpha|}\partial^{\alpha}, \tag{4}$$

and let $P_m(x,\xi)=\sum_{|\alpha|+|\beta|=m} C_{\alpha\beta}x^{\beta}\xi^{\alpha}$ its principal symbol.

Definition 1.3. *Let $\left(\tilde{M}_p\right)$ a sequence satisfying the conditions $(1)-(2)$, we call Roumieu Gelfand-Shilov vector of the operator P associated to $\left(\tilde{M}_p\right)$, every function $u\in C^{\infty}(\mathbb{R}^n)$ such that,*

$$\exists C>0,\ \left\|P^l u\right\|_{L^2(\mathbb{R}^n)}\leq C^{l+1}\tilde{M}_{lm},\ \forall l\in\mathbb{N} \tag{5}$$

For $l=0$ one admits $P^0u=u$.

The space of all Roumieu Gelfand-Shilov vectors of P, is denoted $S^{\{\tilde{M}\}}(\mathbb{R}^n,P)$.

Definition 1.4. *We say that P is globally elliptic if*

$$P_m(x,\xi)\neq 0,\ \forall (x,\xi)\neq(0,0)$$

The iterate property for the globally elliptic operator P in the anisotropic Roumieu Gelfand-Shilov space $S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$, means the inclusion

$$S^{\{\tilde{M}\}}(\mathbb{R}^n,P)\subset S^{\{M\}}_{\{N\}}(\mathbb{R}^n).$$

2. The main result

Theorem 2.1. *Let (M_p), $\left(\tilde{M}_p\right)$ and (N_p) three sequences satisfying the conditions $(1)-(2)$. If the differential operator $P(x,D)$ of the form (4) is globally elliptic and the sequences (M_p), $\left(\tilde{M}_p\right)$ and (N_p) satisfy the condition*

$$\tilde{M}_{p+q}\lesssim M_pN_q,\ \forall p,q\in\mathbb{N}, \tag{6}$$

then $S^{\{\tilde{M}\}}(\mathbb{R}^n,P)\subset S^{\{M\}}_{\{N\}}(\mathbb{R}^n)$.

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INFINITELY MANY OF WEAK SOLUTIONS FOR P-LAPLACIAN PROBLEM WITH IMPULSIVE EFFECTS

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ABSTRACT. By Fountain theorem we obtain infinitely many weak solutions for a class of elliptic problem for the p-laplacian impulsive differential equation with Dirichlet boundary conditions.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 35J60, 35B30, 35B40.

KEYWORDS AND PHRASES. Impulsive differential equation, weak solution, Fountain theorem.

1. Define the problem

In this paper, we will investigate the existence of weak solutions for the following Dirichlet boundary conditions:

$$\begin{cases} -\left(\rho(x)\left|u'\right|^{p-2}u'\right)' + s(x)\left|u\right|^{p-2}u = f(x,\ u) \quad \text{in } [0,\ T] \\ \triangle\left(\left|u'(x_j)\right|^{p-2}u'(x_j)\right) = I_j\left(u(x_j)\right), \quad j = 1, 2, ..., n \\ u(0) = u(T) = 0 \end{cases} \tag{1.1}$$

where $p > 1,\ T > 0,\ \rho(x),\ s(x) \in L^{\infty}([0,\ T])$ satisfy the conditions

$\operatorname{essinf}_{t\in[0,\ T]}\rho(x) > 0$, $\operatorname{essinf}_{t\in[0,\ T]}s(x) > 0$, $0 = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = T$, and $I_j : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous for every $j = 1,\ 2,\ \cdots,\ n$, $f \in (C([0,\ T]) \times \mathbb{R},\ \mathbb{R})$.

Moreover $\triangle\left(\left|u'(x_j)\right|^{p-2}u'(x_j)\right) = \left|u'\left(x_j^+\right)\right|^{p-2}u'\left(x_j^+\right) - \left|u'\left(x_j^-\right)\right|^{p-2}u'\left(x_j^-\right)$,

where $u'\left(x_j^+\right)$ and $u'\left(x_j^-\right)$ denote the right and left limits, respectively, of $u'(x)$ at $x = x_j$, for $j = 1,\ 2, \ldots, n$.

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Initial value problem for Impulsive Caputo-Hadamard Fractional Differential Equations with Integral Boundary Conditions

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Abstract

In this paper, we establish conditions for the existence solutions of initial value problem impulsive Caputo-Hadamard fractional differential equations with integral boundary conditions. We use Banach contraction theory fixed point,Schauder fixed point theorem and nonlinear alternative of Leray-Schauder. Also, we present an example to illustrate our main results.

This paper deals with the existence and uniqueness of solutions to initial value problems (IVP for short) for impulsive fractional differential equation with integral boundary conditions:

$$ {}^{H}_{C}D^{r}y(t) = f(t, y(t)), \text{ for a.e. } t \in J = [1, T], \ , t \neq t_k, \ k = 1, ..., m, 1 < r \leq 2. \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad ,t = t_k, \quad k = 1, .., m \tag{2}$$

$$\Delta y' \mid_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad ,t = t_k, \quad k = 1, .., m \tag{3}$$

$$y(1) = \int_1^T g(s, y(s))ds, y'(1) = \int_1^T h(s, y(s))ds \tag{4}$$

where ${}^{H}_{c}D^{r}$ is the Caputo-Hadamard fractional derivative f, g and $h : J \times \mathbb{R} \to \mathbb{R}$ are given function, $I_k, \overline{I}_k \mathbb{R} \to \mathbb{R}$, $k = 1, ..., m$ are continuous functions, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim\limits_{\varepsilon \to 0^+} y(t_k + \varepsilon)$ and $y(t_k^-) = \lim\limits_{\varepsilon \to 0^-} y(t_k + \varepsilon)$,and $\Delta y'$ has a similar meaning for $y'(t)$,$1 = t_0 < t_1 < ... < t_m < t_{m+1} = T$.

Key words and phrases: Fractional differential equations;impulses; Caputo-Hadamard fractional derivative;fi
xed point theorem.

LIPSCHITZ OPERATORS WITH AN INTEGRAL REPRESENTATION

KHALED HAMIDI

Abstract. For $1 \leq p < \infty$, the class of p-representable linear operators was introduced in 1986 by Roshdi Khalil studied the definition, properties and same results of coincidences for these operators in [?].In this chapter we introduce the concept of Lipschitz p-representable operators, $(1 \leq p < \infty)$, between a metric space and a Banach space. We represent these mappings by a Bochner integrable function, obtaining in this way a rich factorization theory through the classical Banach spaces $C(K), L_p(\mu, K)$ and $L_\infty(\mu, K)$. Also we show that this type of operators in the theory of composition Banach Lipschitz operator ideal, the relationship between these mappings and some well known Lipschitz operators.

Finally for $p = \infty$, we characterize the Lipschitz ∞-representable mappings by a factorization schema through a compact vectors integral operator.

2010 Mathematics Subject Classification. Primary 47B10, 47L20; Secondary 46B28, 46B03.

Keywords and phrases. Lipschitz function, Arens and Eells space, operators with integral representaion, Lipschitz operators with integral representation, vector measure representation.

1. Introduction and Preliminaries

Let X be a pointed metric space. We denote by $X^{\#}$ the Banach space of all Lipschitz functions $T : X \to \mathbb{R}$ which vanish at 0 under the Lipschitz norm

$$Lip\,(T) = \sup_{x \neq E} \frac{d_E(T(x), T(y))}{d_X(x, E)} : x, y \in X$$

We denote by $\mathcal{F}(X)$ the free Banach space over X [2], i.e., $\mathcal{F}(X)$ is the completion of the space

$$Æ\,(X) = \left\{ \sum_{i=1}^{n} \lambda_i \delta_{x_i, y_i}, (\lambda_i)_{i=1}^{n} \subset \mathbb{R}, (x_i)_{i=1}^{n}, (y_i)_{i=1}^{n} \subset X \right\}$$

($Æ(X)$ the space of Arens and Eells of a metric space X [07]) with the norm

$$\|m\|_{\mathcal{F}(X)} = \inf \left\{ \sum_{j=1}^{n} |\lambda_j|\, d_X\left(x_j, x_j'\right), m = \sum_{i=1}^{n} \lambda_i \delta_{x_i, x_i'} \right\},$$

where the function $\delta_{x,E} : X^{\#} \to \mathbb{R}$ is defined as follows $\delta_{x,E}(f) = f(x) - f(E)$.

We have $\mathcal{F}(X)^* = X^{\#}$. For a general theory of free Banach space see [2].

Let X be a metric space and E be a Banach space, we denote by $Lip_0(X;E)$ the Banach space of all Lipschitz functions $T : X \to E$ such that $T(0) = 0$ with pointwise addition and Lipschitz norm. Note that for any $T \in Lip0(X;E)$ there exists a unique linear map (linearization of T) $T_L : \mathcal{F}(X) \to E$ such that $T_L \circ \delta_X = T$ and $\|T_L\| = Lip\,(T)$, i.e., the following diagram commutes: $T : X \overset{\delta_X}{\to} \mathcal{F}(X) \overset{T_L}{\to} E$.

where δ_X is the canonical embedding so that $\langle \delta_X, f \rangle = \langle \delta_{x,0}, f \rangle = f(x)$ for $f \in X^{\#}$.

field (also called a Boolean algebra) of subsets of Ω [3, III, Definition 1.3]. Given a Banach space E, let $\mathcal{G} : \mathcal{F} \to E$ be a vector measure [1, Definition I.1.1] . The variation of $\mathcal{G}$ is the extended nonnegative function $|\mathcal{G}|$ whose value on a set $M \in \mathcal{F}$ is given by$|\mathcal{G}|\,(M) = \sup_\pi \sum_{A\in\pi} \|\mathcal{G}\,(A)\|$. Where the supremum is taken over all partitions π of M into a finite number of pairwise disjoint members of $\mathcal{F}$.

The semivariation of $\mathcal{G}$ is the extended nonnegative function $\|\mathcal{G}\|$ whose value on a set $M \in \mathcal{F}$ is given by $\|\mathcal{G}\|\,(M) = \sup\{|e^* \circ \mathcal{G}|\}\,, e^* \in E^*, \|e^*\| \leq 1$.where $|e^* \circ \mathcal{G}|$ is the variation of the scalar-valued measure $e^* \circ \mathcal{G}$ [1, Definition I.1.4].

2. Definition and factorization of Lipschitz operators with an integral representation

In this section we introduce the Lipschitz operators which admit an integral representation and study their factorization properties

Definition 1. *We say that an operator $T \in Lip_0(X, E)$ admits an integral representation*

$$T(x) = \int_{B_{X^\#}} f(x) d\mathcal{G}\,(f) \qquad (x \in X)$$

for some E^{**}-valued measure $\mathcal{G}$ defined on the Borel sets of $B_{X^\#}$ such that conditions (a) and (b) of $Theorem 1.1$ are verified when we take $K = B_{X^\#}$.

We denote by $\mathcal{IR}^{Lip}\,(X, E)$ the space of all operators $T \in Lip_0(X, E)$ that admit an integral representation. For every $T \in \mathcal{IR}^{Lip}\,(X, E)$, we define $\|T\|_{\mathcal{IR}^{Lip}} = \inf \|\mathcal{G}\|\,(B_{X^\#})$.

Proposition 2.1. *An operator $T \in Lip_0(X, E)$ admits an integral representation if and only if it has an extension $S \in \mathcal{L}(C(B_{X^\#}), E)$, such that the following diagram is commutative: $T : X \overset{i_X}{\to} C(B_{X^\#}) \overset{S}{\to} E$.*

Theorem 2.2. *Given an operator $T \in Lip(X, E)$, the following assertions are equivalent:*

(a) $T \in \mathcal{IR}^{Lip}\,(X, E)$

(b) there is an operator $S : C(B_{X^\#}) \to E$ such that T factors: $T : X \overset{i_X}{\to} C(B_{X^\#}) \overset{S}{\to} E$

(c) there are a compact Hausdorff space K, an embedding $i'_X \in Lip(X, C(K))$, and an operator $S' \in \mathcal{L}(C(K), E)$ such that the following diagram is commutative $T : X \overset{i'_X}{\to} C(B_{X^\#}) \overset{S'}{\to} E$. If one (and then all) of these assertions holds, we have

$$\|T\|_{\mathcal{IR}^{Lip}} = \inf \|S\| = \inf Lip\left(i_X^{'}\right)\left\|S^{'}\right\|.$$

Theorem 2.3. *The pair* $(\mathcal{IR}^{Lip}, \|.\|_{\mathcal{IR}^{Lip}})$ *is a Lipschitz operator ideal*

We devoted to giving applications of the ideal $\mathcal{IR}^{Lip}$ to characterize $\mathcal{L}_\infty$-spaces.

We have proved that $T \in Lip(X, E)$ belongs to $\mathcal{IR}^{Lip}(X, E)$ if and only if T factors through a $L_\infty(\Omega, \mu)$ space, where (Ω, Σ, μ) is a σ-finite measure space, is a stronger property as the following well-known result shows.

Theorem 2.4. *Given an operator* $T \in Lip_0(X, E)$*, consider the following assertions:*

(a) $T \in \mathcal{IR}^{Lip}(X, E)$
(b) $k_E \circ T$ *is extendible;*
(c) $k_E \circ T$ *factors through an* $L_\infty(\Omega, \mu)$*-space;*
(d) $k_E \circ T \in \mathcal{IR}(X, \left(E^{\#}\right)^*)$;
Then $(a) \Longrightarrow (b) \Longleftrightarrow (c) \Longleftrightarrow (d)$*, but* (b) *does not imply* (a)*.*

3. Conclusion

In this work, we have made a study about the Lipschitz operator with an integral representation and we given the operator ideal and some propositions.

In our research we dealt with the Lipschitz opeartor with an integral representation It turns out that an operator belongs to this class if and only if it factors through a $C(K)$ space. As an application, we characterize $\mathcal{L}_\infty$-spaces.

Finally, The propositions of relationship between our class and Lipschitz p-summing operators, Lipschitz p-Grothendieck- integral operators, strongly Lipschitz p-nuclear operators and Lipschitz weakly compact operators are true?.

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LAPLACE-LIKE TRANSFORM HOMOTOPY PERTURBATION METHOD

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Abstract. The main objective of this present work is to combine the homotopy perturbation method with the Shehu transform (also called Laplace-Like tansform) to solve non-linear partial differential equations. The resulting method is called the Shehu Homotopy Perturbation Method (SHPM).

2010 Mathematics Subject Classification. 44A05, 26A33, 44A20, 34K37.

Keywords and phrases. Homotopy perturbation method, Shehu transform method, partial differential equations.

1. Define the Problem

The Shehu transform [5] of the function $v(t)$ of exponential order is defined over the set of functions,

(1)
$$A = \left\{ v(t) : \exists N,\ k_1, k_2 > 0,\ \ |v(t)| < N \exp\left(\frac{|t|}{k_i}\right),\ if\ t \in (-1)^j \times [0,\ \infty) \right\},$$

by the following integral

$$\begin{aligned} \mathbb{H}\left[v(t)\right] &= V(s,u) = \int_0^\infty \exp(\frac{-st}{u}) v(t) dt \\ &= \lim_{\alpha \longrightarrow \infty} \int_0^\alpha \exp(\frac{-st}{u}) v(t) dt,\ s > 0, u > 0. \end{aligned} \tag{2}$$

It converges if the limit of the integral exists, and diverges if not.

The inverse Shehu transform given by

$$v(t) = \mathbb{H}^{-1}\left[V(s,u)\right] = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{1}{u} \exp(\frac{st}{u}) V(s,u) ds, \tag{3}$$

The basic idea of this method is to solve the following general non-linear partial differential equation

$$\frac{\partial^m U(x,t)}{\partial t^m} + RU(x,t) + NU(x,t) = g(x,t), \tag{4}$$

where $\frac{\partial^m U(x,t)}{\partial t^m}$ is the partial derivative of the function $U(x,t)$ of order m $(m = 1, 2, 3)$, R is the linear differential operator, N represents the general non-linear differential operator, and $g(x,t)$ is the source term.

By Applying the Shehu transform on both sides of Equ.(4), and using its properties[1, 2], we get:

(5)
$$\mathbb{H}\left[U(x,t)\right]=\sum_{k=0}^{m-1}\left(\frac{u}{s}\right)^{k+1}\frac{\partial^k U(x,0)}{\partial t^k}+\frac{u^m}{s^m}\mathbb{H}\left[g(x,t)\right]-\frac{u^m}{s^m}\mathbb{H}\left[RU(x,t)+NU(x,t)\right].$$

Applying the inverse transform on both sides of Equ.(5), we get:

$$U(x,t)=G(x,t)-\mathbb{H}^{-1}\left(\frac{u^m}{s^m}\mathbb{H}\left[RU(x,t)+NU(x,t)\right]\right), \tag{6}$$

where $G(x,t)$, represents the term arising from the source term and the prescribed initial conditions.

The classical homotopy perturbation technique HPM for Eq.(6) is constructed as follows [3, 4]:

The solution can be expressed by the infinite series given below

$$U(x,t)=\sum_{n=0}^{\infty}p^nU_n(x,t), \tag{7}$$

where p is considered as a small parameter ($p\in[0,1]$). The non-linear term can be decomposed as:

$$NU(x,t)=\sum_{n=0}^{\infty}p^nH_n(u), \tag{8}$$

where H_n are He's polynomials of U_0, U_1, U_2, ..., U_n, which can be calculated by the following formula

$$H_n(u_0,...,u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}p^iU_i\right)\right]_{p=0}, \quad n=0,1,2,3,\cdots \tag{9}$$

Substituting (7) and (8) in Eq.(6) and using HPM by He, we get:

$$\begin{aligned}\sum_{n=0}^{\infty}p^nU_n&=G(x,t)\\ &-p\left(\mathbb{H}^{-1}\left[\left(\frac{u}{s}\right)^m\mathbb{H}\left[R\sum_{n=0}^{\infty}p^nU_n+\sum_{n=0}^{\infty}p^nH_n(u)\right]\right]\right),\end{aligned} \tag{10}$$

comparing the coeffcients of powers of p; yields

$$\begin{aligned}p^0 &: \quad U_0(x,t)=G(x,t),\\ p^n &: \quad U_n(x,t)=-\mathbb{H}^{-1}\left(\left(\frac{u}{s}\right)^m\mathbb{H}\left[RU_{n-1}(x,t)+H_{n-1}(u)\right]\right),\end{aligned}$$

where $n>0$, $n\in\mathbb{N}$.

Finally, we approximate the analytical solution, $U(x,t)$, by

$$U(x,t)=\lim_{N\to\infty}\sum_{n=0}^{N}U_n(x,t). \tag{11}$$

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Order of Meromorphic Solutions to Non-Homogeneous Linear Differential-Difference Equations

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Abstract. In this paper, we investigate the growth of meromorphic solutions of non-homogeneous linear difference equation

$$A_n(z)f(z+c_n)+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=A_{n+1}(z),$$

where $A_{n+1}(z),\cdots,A_0(z)$ are (entire) or meromorphic functions and c_j $(1,\cdots,n)$ are non-zero distinct complex numbers. Under some conditions on the (lower) order and the (lower) type of the coefficients, we obtain estimates on the lower bound of the order of meromorphic solutions of the above equation. We extend early results due to Chen and Zheng.

Key words: Linear difference equation, Meromorphic solution, Order, Type, Lower order, Lower type

1 Introduction and statement of main results

Throughout this paper, we use the standard notation and basic results of Nivanlinna's value distribution theory. In addition, we use $\rho(f), \mu(f), \tau(f), \underline{\tau}(f)$ to denote respectively the order, the lower order, the type, and the lower type of a meromorphic function f in the complex plane, also when f is entire function we use the notation $\tau_M, \underline{\tau}_M(f)$ respectively for the type and lower type of f (see e.g. ([1] , [2] , [3]), [5]))

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Recently, many articles focused on complex difference equations ([3] , [4] , [5] . The back-ground for these studies lies in the recent difference counterparts of Nevanlinna theory. The key result here is the difference analogue of the lemma on the logarithmic derivative obtained by Halburd-Korhonen [6] and Chiang-Feng [4], independently. Several authors have investigated the properties of meromorphic solutions of complex linear difference equation

$$A_n f(z+c_n) + A_{n-1} f(z+c_{n-1}) + \cdots + A_1 f(z+c_1) + A_0 f(z) = 0 \quad (1)$$

when one coefficient has maximal order or among coefficients having the maximal order, exactly one has its type stricly greater than others and achieved some important results (see e.g. ([3]). Very recently [5], Luo and Zheng have studied the growth of meromorphic solutions of (1) when more than one coefficient has maximal lower order and the lower type strictly greater than the type of other coefficients, and obtained that every meromorphic solution $f \not\equiv 0$ of (1) satisfies $\rho(f) \geq \mu(A_l) + 1$.

OUR POSITION

Question 1.1 What can be said about the growth of meromorphic solutions of non-homogeneous linear difference equation ?

The purpose of this paper is to extend the results of Luo and Zheng for the complex non-homogeneous linear difference equation

$$A_n(z) f(z+c_n) + \cdots + A_1(z) f(z+c_1) + A_0(z) f(z) = A_{n+1}(z), \quad (2)$$

We obtained that every meromorphic solution f of (2) satisfies $\rho(f) \geq \mu(A_l)$ if $A_{n+1} \not\equiv 0$. Furthermore, if $A_{n+1} \equiv 0$, then every meromorphic solution $f \not\equiv 0$ of (2) satisfies $\rho(f) \geq \mu(A_l) + 1$.

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Mild solution of semilinear time fractional reaction diffusion equations with almost sectorial operators and application

Abstract

This study concerns the semilinear reaction diffusion equation involving the Caputo fractional time derivative of order $\alpha(0 < \alpha < 1)$ under some conditions on the initial data and boundary conditions. We prove the existence and uniqueness of a mild solution of abstract fractional Cauchy problems with almost sectorial operators A, By constructing a pair of families of operators in terms of the generalized Mittag-Leffler-type functions and the resolvent operators associated with A. Application part are considered to our problem in space of Hölder continuous functions.

AMS SC 2010: 35K57, 34A08, 65J08, 47A10 , 33E12, 26A33.

Stability results by Krasnoselskii's fixed point theorem for fractional differential problem with initial conditions

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Abstract: A Caputo fractional differential equation with initial conditions is considered. Using Krasnoselskii's fixed point theorem to proof the stability results on a weighted Banach space, then we give an example to illustrate our stability results.

Keywords: Stability and Asymptotically stability , Caputo fractional derivative, Krasnoselskii's fixed point, weighted Banach space.
2010 Mathematics Subject Classification: 34A08, 26A33, 34K20, 34K40

Position of the problem :

Let consider the following IVP of fractional differential equation

$$\begin{cases} {}^{C}D^{p}_{0^+}x(t) = g(t,x(t)) + {}^{C}D^{p-1}_{0^+}f(t,x(t)), \quad t \in [0,+\infty), \\ \\ x(0) = x_0, \quad x'(0) = x_1. \end{cases} \tag{0.1}$$

where $1 < p < 2, (x_0, x_1) \in \mathbb{R}^2, f, g : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions with $f(t,0) = g(t,0) \equiv 0$ and ${}^{C}D^{p}$ is the standard Caputo fractional derivative of order p.

In this presentation, we will show the stability result of the solution in a weighted Banach space, by the Krasnoselskii's fixed point theorem.

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A class of autonomous differential systems with explicit limit cycles

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Abstract

For a given family of planar differential equations it is a very difficult problem to determine an upper bound for the number of its limit cycles. In this paper we give a family of planar polynomial differential systems of degree odd whose limit cycles can be explicitly described using polar coordinates. The given family of planar polynomial differential systems can have at most two explicit limit cycles, one of them algebraic and the other one non–algebraic.

2010 Mathematics Subject Classification: 34A05, 34C07.
Key Words: Planar polynomial differential system, algebraic and non–algebraic limit cycle, hyperbolicity, Riccati differential equation.

1 Main result

As a main result, we shall prove the following theorem.

Theorem 1 *The three-parameters polynomial differential system*

$$\begin{cases} \dot{x} = \left(\gamma x - x\left(x^2+y^2\right)^2 - 4\gamma y\right)\left(a\left(x^2+y^2\right)^2 + 4bxy\left(x^2-y^2\right)\right) - x\left(\left(x^2+y^2\right)^2-\gamma\right)^2 \\ \dot{y} = \left(\gamma y - y\left(x^2+y^2\right)^2 + 4\gamma x\right)\left(a\left(x^2+y^2\right)^2 + 4bxy\left(x^2-y^2\right)\right) - y\left(\left(x^2+y^2\right)^2-\gamma\right)^2 \end{cases} \tag{1}$$

where $a, b, \gamma \in \mathbb{R}_+^*$ possesses exactly two limit cycles: the circle $(\Gamma_1) : \left(x^2+y^2\right)^2 - \gamma = 0$ surrounding a transcendental limit cycle (Γ_2) explicitly given in polar coordinates (r, θ) by the equation

$$r = \left(\gamma + \gamma \frac{e^{-\theta}}{\frac{r_*^4}{r_*^4-\gamma} - e^{-\theta} + f(\theta)}\right)^{\frac{1}{4}},$$

with $f(\theta) = \int_0^\theta \frac{e^{-s}}{a+b\sin 4s}ds$ and $r_* = \left(\gamma \frac{f(2\pi)}{1-e^{-2\pi}+f(2\pi)}\right)^{\frac{1}{4}}$, when the following condition is assumed :

$$b^2 < a^2.$$

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Example 2 *For $a = 2$, $b = \frac{1}{2}$, and $\gamma = 3$ the system* (1) *becomes*

$$\begin{cases} \dot{x} = 2\left(3x - x\left(x^2+y^2\right)^2 - 12y\right)\left(\left(x^2+y^2\right)^2 + xy\left(x^2-y^2\right)\right) - x\left(\left(x^2+y^2\right)^2 - 3\right)^2 \\ \dot{y} = 2\left(3y - y\left(x^2+y^2\right)^2 + 12x\right)\left(\left(x^2+y^2\right)^2 + xy\left(x^2-y^2\right)\right) - y\left(\left(x^2+y^2\right)^2 - 3\right)^2 \end{cases}$$

his system possesses two limit cycles : the circle (Γ_1) : $\left(x^2+y^2\right)^2 - 3 = 0$ *surrounding a transcendental limit cycle* (Γ_2) *explicitly given in polar coordinates* (r, θ) *by the equation*

$$r = \left(2 + 2\frac{e^{-\theta}}{\frac{r_*^4}{r_*^4-\gamma} - e^{-\theta} + f(\theta)}\right)^{\frac{1}{4}}$$

with $f(\theta) = \displaystyle\int_0^\theta \frac{e^{-s}}{2+\frac{1}{2}\sin 4s}ds$ *and* $r_* = \left(3\frac{f(2\pi)}{1-e^{-2\pi}+f(2\pi)}\right)^{\frac{1}{4}} = 0.995\,07.$

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NON-EXTINCTION OF SOLUTIONS FOR A CLASS OF P-LAPLACIAN NONLOCAL HEAT EQUATIONS WITH LOGARITHMIC NONLINEARITY

TOUALBIA SARRA

ABSTRACT. problem of a nonlocal heat equations with logarithmic nonlinearity in a bounded domain. By using the logarithmic Sobolev inequality and potential wells method, we obtain the decay, blow-up and non-extinction of solutions under some conditions, and the results extend the results of a recent paper Lijun Yan and Zuodong Yang (2018)..

1. INTRODUCTION

In this paper, we consider the Neumann problem to the following initial parabolic equation with logarithmic source:

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p-2}u\log|u| - \oint_\Omega |u|^{p-2}u\log|u|\,dx, & x\in\Omega, t>0,\\ \dfrac{\partial u(x,t)}{\partial\eta} = 0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0, & x\in\Omega, t>0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p\in(2,+\infty)$, $\oint_\Omega u_0dx = \frac{1}{|\Omega|}\int_\Omega u_0dx = 0$ with $u_0\neq 0$.

For prouve our result It is necessary to note that the presence of the logarithmic nonlinearity causes some difficulties in deploying the potential well method. In order to handle this situation we need the following logarithmic Sobolev inequality which was introduced in.

Lemma 1.1. *Let $p>1, \mu>0$, and $u\in W^{1,p}(\mathbb{R}^n)\setminus\{0\}$. Then we have*

$$p\int_{\mathbb{R}^n}|u(x)|^p\log\left(\frac{|u(x)|}{\|u(x)\|_{L^p(\mathbb{R}^n)}}\right)dx + \frac{n}{p}\log\left(\frac{p\mu e}{n\mathcal{L}_p}\right)\int_{\mathbb{R}^n}|u(x)|^p\,dx \le \mu\int_{\mathbb{R}^n}|\nabla u(x)|^p\,dx,$$

where

$$\mathcal{L}_p = \frac{p}{n}\left(\frac{p-1}{e}\right)^{p-1}\pi^{-\frac{p}{2}}\left[\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n\frac{p-1}{p}+1\right)}\right]^{\frac{p}{n}}.$$

Non-extinct in finite time

1991 *Mathematics Subject Classification.*

Key words and phrases. non-extinction in finit time, heat equation, logarithmic Sobolev inequality.

Definition: (Finite time blow-up) Let $u(x,t)$ be a weak solution of (1).We call $u(x,t)$ blow-up in finite time if the maximal existence time T is finite and

$$\lim_{t\to T^-} \|u(.,t)\|_2 = +\infty.$$

Lemma 1.2. *Let ϕ be a positive, twice differentiable function satisfying the following conditions*

$$\phi(\bar{t}) > 0, \ \text{and} \ \ \phi^{'}(\bar{t}) > 0,$$

for some $\bar{t} \in [0,T)$, and the inequality

$$\phi(t)\phi^{''}(t) - \alpha(\phi^{'}(t))^2 \geq 0, \quad \forall t \in \left[\bar{t}, T\right], \tag{1.2}$$

where $\alpha > 1$. Then we have

$$\phi(t) \geq \left(\frac{1}{\phi^{1-\alpha}(\bar{t}) - \sigma(t-\bar{t})}\right), \quad t \in \left[\bar{t}, T^*\right).$$

with σ is a positive constant, and

$$T^* = \bar{t} + \frac{\phi(t)}{(\alpha-1)\phi'(\bar{t})}.$$

This implies

$$\lim_{t\to T^*} \phi(t) = \infty.$$

Theorem 1.3. *Assume $0 < J(u_0) < M$ and $u \in W_1^-$, then the solution $u(x,t)$ of problem (1) is non-extinct in finite time, defined by*

$$T^* = \bar{t} + \frac{\int_0^t \|u(s)\|_2^2 \, ds}{(\frac{p-2}{2}) \left\|u(\bar{t})\right\|_2^2}, \ s \in \left[\bar{t}, T^*\right).$$

2.

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NONLINEAR VOLTERRA INTEGRAL EQUATIONS AND THEIR SOLUTIONS

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Abstract. In order to solve nonlinear Volterra integral equations with weakly singular kernels, we need to convert these integral equations into nonlinear systems (see [1, 2, 3, 4, 5, 7] for details of integral equations). For that, we require a product integration method which leads to attain best approximate solutions (see [6, 8, 9]). By giving a numerical application, we can prove that we have precise approximate solutions.

2010 Mathematics Subject Classification.

Keywords and phrases. Nonlinear integral equations, Volterra equations, Weakly singular kernels.

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NONLINEAR ANISOTROPIC ELLIPTIC UNILATERAL PROBLEMS WITH VARIABLE EXPONENTS AND DEGENERATE COERCIVITY

HOCINE AYADI

Abstract. In this talk, we prove the existence of entropy solutions for some nonlinear anisotropic degenerate elliptic unilateral problems with L^1-data. The functional framework involves anisotropic Sobolev spaces with variable exponents as well as variable exponent Marcinkiewicz spaces. Our results are natural generalization and extension of previous studies [1, 2, 3].

2010 Mathematics Subject Classification. 35J87, 35J70, 35D99.

Keywords and phrases. Unilateral problems, entropy solutions, variable exponents, degenerate coercivity, L^1-data.

1. Define the problem

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$ and $f \in L^1(\Omega)$. We consider the following nonlinear anisotropic problem

$$\begin{cases} -\sum_{i=1}^{N} D_i\left(\frac{a_i(x,\nabla u)}{(1+|u|)^{\gamma_i(x)}}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $p_i : \overline{\Omega} \to (1, +\infty)$ and $\gamma_i : \overline{\Omega} \to [0, +\infty)$ for $i = 1, \dots, N$ are continuous functions such that

$$1 < \overline{p}(x) < N \quad \text{for all } x \in \overline{\Omega}. \tag{2}$$

and

$$p_+(x) < \overline{p}^*(x) \quad \text{for all } x \in \overline{\Omega}, \tag{3}$$

where $\frac{1}{\overline{p}(x)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i(x)}$, $p_+(x) = \max_{1\leq i\leq N}\{p_i(x)\}$, and $\overline{p}^\star(x) = \frac{N\overline{p}(x)}{N-\overline{p}(x)}$.

We assume, for $i = 1, \dots, N$, that $a_i : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying for almost every $x \in \Omega$ and for every $\xi = (\xi_1, \dots, \xi_N)$, $\xi' = (\xi'_1, \dots, \xi'_N) \in \mathbb{R}^N$, with $\xi_i \neq \xi'_i$, the following assumptions

$$|a_i(x,\xi)| \leq \beta|\xi_i|^{p_i(x)-1}, \tag{4}$$

$$a_i(x,\xi)\xi_i \geq \alpha|\xi_i|^{p_i(x)}, \tag{5}$$

$$\left[a_i(x,\xi) - a_i(x,\xi')\right]\left[\xi_i - \xi'_i\right] > 0, \tag{6}$$

where $\alpha > 0$, and $\beta > 0$.

we denote

$$L^\infty_+(\Omega) = \{h : \Omega \to \mathbb{R} \text{ is measurable} : \ 0 < h^- \leq h^+ < \infty\},$$

where

$$h^- = \operatorname*{ess\,inf}_{x\in\Omega} h(x) \text{ and } h^+ = \operatorname*{ess\,sup}_{x\in\Omega} h(x).$$

Definition 1.1. *Let $p \in L^\infty_+(\Omega)$. We say that a measurable function $u : \Omega \to \mathbb{R}$ belongs to the Marcinkiewicz space $\mathcal{M}^{p(\cdot)}(\Omega)$ if*

$$\int_{\{|u|>\lambda\}} \lambda^{p(x)} dx \leq C, \quad \textit{for all } \lambda > 0.$$

where χ_E denotes the characteristic function of a measurable set E.

Let $k \geq 0$, we consider the usual truncation $T_k(s)$ defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k\frac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$

For a given function $\psi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we define the following convex set

$$\mathcal{K}_\psi = \left\{ v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) : v \geq \psi \text{ a.e. in } \Omega \right\}.$$

Definition 1.2. *An entropy solution of the obstacle problem $(\mathcal{A}, f, \psi)$ associated to the problem (1) is a measurable function u such that*

$$\begin{cases} u \geq \psi \ \textit{a.e. in}\, \Omega, \\ T_k(u) \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \ \forall k > 0, \\ \displaystyle\sum_{i=1}^N \int_\Omega \frac{a_i(x,\nabla u) D_i T_k(u-v)}{(1+|u|)^{\gamma_i(x)}} dx \leq \int_\Omega f\, T_k(u-v) dx, \\ \forall v \in \mathcal{K}_\psi \cap L^\infty(\Omega). \end{cases} \tag{7}$$

Our main result is the following theorem.

Theorem 1.1. *Assume that (4)-(6), and (3) hold true and that*

$$0 \leq \gamma_+^+ < p_-^- - 1, \tag{8}$$

where $\gamma_+^+ = \max\{\gamma_1^+, \dots, \gamma_N^+\}$ and $p_-^- = \min\{p_1^-, \dots, p_N^-\}$. Then, the problem (1) has at least one entropy solution $u \in \mathcal{M}^{q(\cdot)}(\Omega)$ and $|D_i u|^{\xi_i(x)} \in \mathcal{M}^{q(\cdot)}(\Omega)$, $i = 1, \dots, N$, with

$$q(x) = p_+(x)\left(1 - \frac{1+\gamma_+^+}{p_-^-}\right) \ \textit{and}\ \xi_i(x) = \frac{p_i(x)}{q(x)+1+\gamma_i(x)}.$$

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ON EXACT CONTROLLABILITY AND COMPLETE STABILIZABILITY FOR DEGENERATE SYSTEMS IN HILBERT SPACES

MOHAMED HARIRI AND MEHDI BENABDALLAH

Abstract. The aim of this research is concerned with the relations between exact controllability and complete stabilizability for degenerate systems in Hilbert spaces. Using the spectral theory of the operator pencil $\lambda A - B$, $\lambda \in \mathbb{C}$ to obtain some necessary and sufficient condition for the exact controllability. Where A and B are bounded operators in Hilbert spaces, the operator A is not necessarily invertible.

2010 Mathematics Subject Classification. 34L05, 93B05, 93D20.

Keywords and phrases. Spectral theory, exact controllability, stabilizability of control systems.

1. Introduction

In the present paper, we consider an control problem for the system described by the degenerate differential equation

$$Ax'(t) = Bx(t) + Cu(t) \quad , \quad t \geq 0, \quad x \in \mathcal{H} \quad . \tag{1}$$

For system (1) we pose the initial condition

$$x(t_0) = x_0. \tag{2}$$

Where A , B and C are bounded operators in Hilbert spaces $\mathcal{H}$. The operator A is not necessarily invertible, the function u is square integrable in the sense of Bochner.

The linear part of system (1) corresponds to the operator pencil

$$L(\lambda) = \lambda A - B \ , \ \lambda \in \mathbb{C}$$

which is defined on the set $D = D_A \cap D_B \neq \{0\}$ we denote the space of bounded linear operators mapping $\mathcal{H}$ into $\mathcal{H}$, we use the resolvent

$$R(\lambda) = L^{-1}(\lambda).$$

For a detailed expositions, see [6, 7, 8] we have the direct sum decompositions $\mathcal{H} = \mathcal{H} \dot{+} \mathcal{H}$, $D = D_1 \dot{+} D_2$,

such that the operator pencil $L(\lambda)$ has the block structure

$$\lambda A - B = diag[\lambda A_1 - B_1, B_2], \qquad D_1 \dot{+} D_2 \to \mathcal{H} \dot{+} \mathcal{H}.$$

If $D_1 \neq \{0\}$, then there exists an inverse operator A_1^{-1}.

The mild solution of the system (1)

$$x(t, x_0, u) = S(t)x_0 + \int_0^t S(t-\tau)A_1^{-1}Cu(\tau)d\tau, \tag{3}$$

Definition 1.1. *The system* (1) *is exactly controllable with respect to* $L^2([0,1],\mathcal{H})$ *on* $[0,t]$ *such that for all* $x_0, x_1 \in \mathcal{H}$ *and for some control* $u(t)$*, we have* $x(t, x_0, u) = x_1$.

Consider the following bounded linear operators

$$G : L^2([0,1],\mathcal{H}) \to \mathcal{H}, \qquad Gx = \int_0^t S(t-\tau)A_1^{-1}Cu(\tau)d\tau$$

$$W : \mathcal{H} \to \mathcal{H}, \qquad Wx = \int_0^t S(t-\tau)C^*A_1^{-1}CS^*(t-\tau)xd\tau$$

is a uniformly positive definite operator and then invertible, i.e W^{-1}

Definition 1.2. *The system* (1) *is completely stabilizable if for all* $\alpha \in \mathbb{R}$ *there exists a linear bounded operator* $F \in \mathcal{L}(\mathcal{H},\mathcal{H})$ *and constant* $M > 0$ *such that the semi-group generated by* $(B_1A_1^{-1} + CF, D_1 \dotplus D_2)$ *say* $S_F(t)$*, verifies:*

$$||S_F(t)|| \leq Me^{\alpha t}, \quad t \geq 0.$$

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ON SADOVESKII FIXED POINT THEOREMS UNDER THE INTERIOR CONDITION IN TERMS OF WEAK TOPOLOGY

AHMED BOUDAOUI AND NOURA LAKSACI

Abstract. In this work, the authors focus to give an extension of Sadoveskii's fixed point theorem for non-self mappings. These mappings are satisfying the so-called interior condition. The main assumptions of the results are formulated in the weak topology settings of Banach spaces, and Deblasi measure of weak noncompactness.

2010 Mathematics Subject Classification. 47H10, 54H25, 47H08.

Keywords and phrases. Fixed point theorems, λ-set weak contractive, measure of weak noncompactness, Interior condition, Minkowski functional.

1. Define the problem

The problems of the existence solutions in functional analysis may be transformed to fixed point problem of the form

$$N\varrho = \varrho \qquad \varrho \in \mathcal{K} \subset \mathcal{E},$$

with $\mathcal{K}$ has some topological and geometrical hypotheses, $\mathcal{E}$ is a Banach spaces and N is a nonlinear operator. For this, many researchers have been interested in the case where the Banach space is endowed with its norm topology; however, others have been focused in the case where the Banach space equipped under its weak topology setting. The history of fixed point theory in Banach space equipped with its weak topology was started by Tychonoff in 1935 as follow:

Theorem 1.1. [6] *Let $\mathcal{E}$ be a Banach space and let $\mathcal{K}$ be a weakly compact, convex subset of $\mathcal{E}$. If the mapping $N : \mathcal{K} \to \mathcal{K}$ is weakly continuous, then it has a fixed point.*

In 1997, Banaś [1] discussed the case where the operator is not necessary weakly compact, and gave an analogue result of Theorem1.1. Actually, he used the concept of λ-set weakly contraction and weakly condensing with respect to a measure of weak noncompactness.
In some cases, we deal with operators which are continuous and weakly compact. Since neither the continuity implies the weak continuity nor the weak compactness implies the strong compactness, we can't use Schauder or Tychonoff's fixed point theorems, for this reason Latrach et al. [4] introduced the following condition:

$$(\mathcal{A}1) \quad \begin{cases} \text{If } (\varrho_n)_{n\in\mathbb{N}} \text{ is a weakly convergent sequence in } \mathcal{E}, \text{ then} \\ (N\varrho_n)_{n\in\mathbb{N}} \text{ has a strongly convergent subsequence in } \mathcal{E}. \end{cases}$$

Theorem 1.2. [4] *Let $\mathcal{K}$ be a nonempty closed convex subset of a Banach space $\mathcal{E}$. Assume that $N : \mathcal{K} \to \mathcal{K}$ is a continuous map which verifies $(\mathcal{A}1)$. If N is weakly condensing, then there exists $\varrho \in \mathcal{K}$ such that $N\varrho = \varrho$.*

The authors in [5] established the following nonlinear alternative version fixed point theorem of Theorem1.2:

Theorem 1.3. [5] Let C be a nonempty, closed convex subset of $\mathcal{E}$ and $\mathcal{K} \subseteq C$ an open set (with the topology of C) and let z be an element of $\mathcal{K}$. Assume $N : \bar{\mathcal{K}} \longrightarrow C$ is a continuous weakly-condensing which satisfies $(\mathcal{A}1)$. If $N(\bar{\mathcal{K}})$ is bounded, then either
(a) N has a fixed point, or
(b) there exist a point $\varrho \in \partial_C \mathcal{K}$ (the boundary of $\mathcal{K}$ in C) and $\lambda \in]0, 1[$ with $\varrho = \lambda N(\varrho) + (1 - \lambda)z$.

Therefore, in this study we are looking to prove the fixed point result for non self weakly condensing mapping, fulfills the following so-called Interior condition. This latter signify that there is $\delta > 0$ such that

$$N(\varrho) \neq \beta\varrho \text{ for } \varrho \in \mathcal{K}_\delta,\ \beta > 1 \text{ and } N(\varrho) \notin \bar{\mathcal{K}}, \qquad (\mathcal{IC})$$

where $\mathcal{K}_\delta = \{\varrho \in \mathcal{K} : \operatorname{dist}(\varrho, \partial\mathcal{K}) < \delta\}$. This condition was mentioned in the first time in the following articles [2, 3].

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ON SOME PROPERTIES OF NUCLEAR POLYNOMIALS

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Abstract. Nuclear polynomials between Banach spaces have been studied since 1983 seminal paper [8] by A. Pietsch. These classes of polynomials have received the attention of many authors. well knew continuous m- homogeneous polynomials forms ($n \geq 2$) can not always extension, as always every continuous linear functional defined over a normed space, these can be extended to any superspace, by the Hahn-Banach Theorem. Extendible polynomials have defined in ([1], [3], [5] ...).
The objective of this note is to study the extensibility of the ideals of polynomials and to demonstrate that the extensible and liftable nuclear polynomials are nuclear, and and present some of the results of the nonlinear theory associated with them.

Keywords. extension property, lifting property, nuclear polynomial.

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ON THE AVERAGE NO-REGRET CONTROL FOR DISTRIBUTED SYSTEMS WITH INCOMPLETE DATA

MOUNA ABDELLI AND ABDELHAK HAFDALLAH

Abstract. We discuss the averaged control of distributed systems depending on an unknown parameter and with incomplete data following the notion of averaged no-regret control. We associate with the averaged no-regret control a sequence of averaged low-regret controls defined by a quadratic perturbation. In the first part, we prove that the perturbed system corresponds to a sequence of standard averaged control problems and converges to the averged no-regret control for which we obtain a singular optimality system. We give also some applications. In the second part, we show how the method can be extended to the evolution case. Equations of parabolic type, Petrowsky type, or hyperbolic type are considered.

2010 Mathematics Subject Classification. 35Q93, 49J20, 93C05. 93C41.

Keywords and phrases. averaged No-regret control, averaged low-regret control, optimality condition, singular optimality system.

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ON THE EXISTENCE OF A WEAK SOLUTION FOR A CLASS OF NONLOCAL ELLIPTIC PROBLEMS

ELMEHDI ZAOUCHE

ABSTRACT. We prove the existence of a weak solution for a class of nonlocal heterogeneous elliptic problems using the Tychonoff fixed point theorem.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 35A05, 35J60, 35J25.

KEYWORDS AND PHRASES. Tychonoff fixed point theorem, nonlocal heterogeneous elliptic problems, weak solution; existence.

1. DEFINITION OF THE PROBLEM

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$. We consider the following weak formulation of nonlocal heterogeneous elliptic problems (see [1]):

$$\text{(1)} \qquad \begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that :} \\ \displaystyle\int_\Omega a(x)\nabla u \cdot \nabla\xi \, dx = \frac{\displaystyle\int_\Omega (g(x,u))^\alpha \xi \, dx}{\left(\displaystyle\int_\Omega g(x,u)\, dx\right)^\beta} \\ \forall \xi \in H_0^1(\Omega), \end{cases}$$

where $a(x) = (a_{ij}(x))_{ij}$ is an $n\times n$ matrix function defined almost everywhere on Ω satisfying for two positive constants λ, Λ,

$$\forall \xi \in \mathbb{R}^n : \quad \lambda|\xi|^2 \leq a(x)\xi \cdot \xi \quad \text{a.e. } x \in \Omega,$$
$$\forall \xi \in \mathbb{R}^n : \quad |a(x)\xi| \leq \Lambda|\xi| \quad \text{a.e. } x \in \Omega,$$

$g : \Omega\times\mathbb{R} \to \mathbb{R}$ is a function such that for all $t \in \mathbb{R}$, $x \mapsto g(x,t)$ is measurable, $t \mapsto g(x,t)$ is continuous for a.e. $x \in \Omega$ and for some function $h \in L^1(\Omega)$,

$$\forall t \in \mathbb{R}, \text{ a.e. } x \in \Omega, \quad 0 < g(x,t) \leq h(x)$$

and α, β satisfy one of the two following assumptions,

$$0 \leq \alpha \leq \frac{1}{2} \quad \text{and} \quad \beta \leq \alpha;$$
$$\alpha > \frac{1}{2}, \quad \beta \leq \frac{1}{2} \quad \text{with} \quad h(x) \leq 1 \quad \text{a.e. } x \in \Omega.$$

Under the hypotheses mentioned above on a, g, α and β, we prove an existence theorem of a weak solution for the problem (1) using the Tychonoff fixed point theorem ([2]).

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ON A FRACTIONAL p-LAPLACIAN PROBLEM WITH DISCONTINUOUS NONLINEARITIES.

HANAÂ ACHOUR[A] AND SABRI BENSID[B]

Abstract. In this paper, we are concerned by the study of a discontinuous elliptic problem involving a fractional p-Laplacian arising in different contexts. Under suitable conditions, we provide the existence and multiplicity result via the nonsmooth critical point theory.

2010 Mathematics Subject Classification. 34R35, 35J25, 35B38.

Keywords and phrases. Discontinuous nonlinearities , free boundaries, fractional p-Laplacian, critical point, variational method.

1. Define the problem

This paper provides a generalization of the fractional elliptic problem, which various authors have recently studied. For a brief panorama of works dealing with this type of problem, see [1–4,6,7,10]. In particular, we extend the author's results in [5] for the fractional p-Laplacian with sign-changing nonlinearities and n discontinuities. More precisely, we are concerned about studying the existence and multiplicity of solutions to the following problem.

$$\begin{cases} (-\Delta)_p^s u = m(x) \sum_{i=1}^{n} H(u-\mu_i) & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $p \in (1,\infty)$, $s \in (0,1)$, $(N > ps)$ with smooth boundary $\partial\Omega$, m is a sign-changing function, H is the Heaviside function, $\mu_i > 0$ is a real parameter verifying $\mu_1 < \mu_2 < \cdots < \mu_n, (n \in \mathbb{N}^*)$ and $(-\Delta)_p^s$ is the fractional p-Laplacian which up to normalization functions may be defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \, dy, \; x \in \mathbb{R}^N,$$

where $B_\varepsilon(x)$ is the open ε-ball of centre x and radius ε. Note that problem (1) can be regarded as a free boundary problem with unknown regions to be characterized, reduced to a boundary value problem.

In the presence of discontinuous nonlinearities, our problem (1) has a variational nature, and its eventual solutions, which solve it in the multivalued sense, can be constructed as the critical points of the following associated Euler-Lagrange functional

$$E(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, dxdy - \int_\Omega m(x) G(u(x)) \, dx,$$

where $G : [0, +\infty[\to \mathbb{R}$ is $G(t) = \sum_{i=1}^{n} \int_0^t H(s - \mu_i)\ ds$.

We remark that the functional E is not Fréchet differentiable, which means that the classical variational methods are not applicable. Therefore, our main objective is setting appropriate assumptions on the functions m and G and using the Chang theory [8] for nondifferentiable functions due to Clarke [9], and we prove that the energy functional E is only locally Lipschitz continuous. Hence, by the nonsmooth mountain pass theorem version, we prove the existence of solutions to the problem (1).

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ON AN EVOLUTION PROBLEM INVOLVING FRACTIONAL DIFFERENTIAL EQUATIONS

SOUMIA SAÏDI

Abstract. We deal in the present work with a system coupled by a differential inclusion involving subdifferential operator and a fractional differential equation.

2010 Mathematics Subject Classification. 34A60, 49J52, 49J53

Keywords and phrases. Differential inclusion, subdifferential operator, fractional derivative.

1. Main result

Differential equations of fractional order have recently been proved valuable tools in modeling many phenomena in various fields of science and engineering. There are many applications to problems in viscoelasticity, electrochemistry, control, porous media, electromagnetics, etc. There has been a significant theoretical development in fractional differential equations in recent years. In particular, the existence of solutions of boundary value problems and boundary conditions for implicit fractional differential equations and integral equations with fractional derivatives constitutes an attractive subject of research.

We investigate here a system involving a differential inclusion and a differential equation with fractional derivatives. In our development, we use an existence and uniqueness result concerning first-order evolution problems with single-valued perturbations to state our main theorem.

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ON GENERAL BITSADZE-SAMARSKII PROBLEMS OF ELLIPTIC TYPE IN L^p CASES

HAMDI BRAHIM, MAINGOT STÉPHANE, AND MEDEGHRI AHMED

ABSTRACT. This work is devoted to the study of General Bitsadze-Samarskii Problems of elliptic type in the framework of UMD Banach spaces. Here, we obtain some results about existence, uniqueness and regularity of the solution. We define two types of solutions (strict and semi-strict solutions) and we give necessary and sufficient conditions on the data to obtain these results.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 34G10, 35J05, 35J15, 35J25, 35J99.

KEYWORDS AND PHRASES. Nonlocal boundary conditions, Analytic semigroups, Bounded imaginary powers of operators, UMD spaces.

1. POSITION OF THE PROBLEM

In this work, we study a non local elliptic problem of Bitsadze-Samarskii type in the framework of L^p spaces.
Let $x_0 \in [0,1[$, we consider:

$$(P1): \begin{cases} -u''(x) + Au(x) = f(x), \text{ a.e. } x \in]0,1[\\ u(0) = u_0, \\ u(1) - Hu'(x_0) = u_{1,x_0}, \end{cases}$$

where $f \in L^p(0,1;X)$, $1 < p < +\infty$, X is a complex Banach space, u_0 and u_{1,x_0} are elements of X. A and H are two closed linear operators in X with domains $\mathcal{D}(A)$ and $\mathcal{D}(H)$, respectively.

Our main goal is to give (under some hypotheses) necessary and sufficient conditions on the data to obtain:

- Semi-strict solution, i.e. u verify (P1) and:

$$u \in W^{2,p}(0,1-\varepsilon;X) \cap L^p(0,1-\varepsilon;\mathcal{D}(A)) \text{ et } u' \in L^p(0,1;X).$$

- Strict solution, i.e. u verify (P1) and:

$$u \in W^{2,p}(0,1;X) \cap L^p(0,1;\mathcal{D}(A)).$$

The method is based essentially on the construction of a representation of the solution, using the semigroupes theory, fractional power of operators, the interpolation spaces and sum operators theory.

We give finally, some resultats concerned the existence, unicity and regularity of the solution of this problem. See Hamdi et al. [1].

Our work completes the one studied by Hammou et al. [2], where the authors considered the same problem (P1), for $x_0 = 0$.

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ON SEMICLASSICAL FOURIER INTEGRAL OPERATORS, SCHRÖDINGER PROPAGATORS AND COHERENT STATES

OUISSAM ELONG

Abstract. We introduce a class of semiclassical Fourier integral operators using Fourier-Bargmann transform. We will show that the propagator defined by the solution of the time dependent Schrdinger equation with subquadratic Hamiltonian $H(t)$ is a semiclassical Fourier integral operator of order 0 associated to the Hamilton flow generated by the classical Hamiltonian $H(t)$.

2010 Mathematics Subject Classification. 35S30, 35Q41.

Keywords and phrases. Semiclassical Fourier integral operators, time dependent Schrdinger equation, Coherent states, Fourier-Bargmann transform.

1. Define the problem

The time-dependent Schrdinger equation is a linear partial differential equation

$$i\bar{h}\partial_t\psi(t) = \hat{H}(t)\psi(t), \quad \psi(t = t_0) = \psi_0, \tag{1}$$

where ψ_0 is an initial state, $\hat{H}(t)$ is the quantum Hamiltonian depending on time t, defined as a continuous family of self-adjoint operators in the Hilbert space $L^2(\mathbb{R}^n)$ and $\hbar > 0$ is the Plank constant.

In this talk we prove that the parametrix of (1) constructed in [2] and its remainder operator are semiclassical FIO of order 0. This work, is the semiclassical version of results obtained in [3] for $\hbar = 1$. Here we control the singular limit $\hbar \searrow 0$.

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On some nonlinear p(x)-elliptic problems with convection term

Hadjira LALILI*

March 7, 2021

Abstract

In this work, we deal with elliptic systems of the form:

$$(\mathcal{P}) \qquad \begin{cases} -\Delta_{p(x)} u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $\Delta_{p(.)} u := \operatorname{div}\left(|\nabla u|^{p(.)-2}\nabla u\right)$ is the so-called $p(.)$-Laplacian operator, Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Under some conditions growth on the nonlinearities, we search solutions for the problem $(\mathcal{P})$ using Fredholm-type result for a couple of nonlinear operators [3].

Key words: $p(x)$-Laplacian; generalized Lebesgue-Sobolev spaces; surjectivity theorem.

Mathematics Subject Classification: 47H05, 47H10.

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- **Title** : On the existence of a shape derivative formula in the Bruun-Minkowski theory
- **Abstract** : In this work, we consider again the shape derivative formula [1] for a volume cost functional which we studied in preceding papers where we used the Minkowski deformation and the support functions in the convex setting. Here, we extend it to some non convex domains, namely the star-shaped ones. The formula also happens to be an extension of a well known formula in the Brunn-Minkowski theory. Finally, we illustrate the formula by applying it to the computation of the shape derivative for a shape optimization problem and by giving an algorithm based on the gradient method.
- **Keywords** : shape optimization, shape derivative, volume functional, convex domain, starshapeddomain, support function, gauge function, Minkowski deformation, Brunn-Minkowski

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ON THE NUMERICAL RANGE OF M ISOMETRY AND QUASI-ISOMETRY OPERATOR

ZAIZ KHAOULA[1] MANSOUR ABDELOUAHAB[2]

Abstract. Let $\mathbb{B}(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. For all $A \in \mathbb{B}(H)$, we define the numerical range $W(A)$ as collection of all complex numbers of the form $\langle Ax, x\rangle$ where $x \in H$. More precisely

$$W(A) = \{\langle Ax, x\rangle : x \in H,\ \|x\| = 1\}.$$

In this work, we study some topological and analytic properties of numerical range of $m-$ isometry operators and quasi-isometry operators, where A is an $m-$isometry if and only if

$$\sum_{k=0}^{m}(-1)^k \left(\begin{array}{c} m \\ k \end{array}\right) \left\|T^k h\right\|^2 = 0$$

for all $h \in H$, and A is a quasi-isometry if $A^{*2}A^2 = A^*A$.

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Key words and phrases. Numerical Range, $m-$ isometry operator, quasi-isometry.

On the positive Cohen p-nuclear m-linear operators

Amar Bougoutaia * [1], Amar Belacel *

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In this talk, we introduce and study the concept of positive Cohen p-nuclear multilinear operators between Banach lattice spaces. We prove a natural analog to the Pietsch domination theorem for this class.

*Speaker

ON THE SPECTRAL BOUNDARY VALUE PROBLEMS AND BOUNDARY APPROXIMATE CONTROLLABILITY OF LINEAR SYSTEMS

NASSIMA KHALDI

ABSTRACT. The main subject of this paper is the study of a general spectral boundary value problems with right invertible (resp. left invertible) operators and corresponding initial boundary operators. The obtained results are used to d escribe the approximate boundary controllability of linear systems in abstract operator-theoretic setting.

2000 MATHEMATICS SUBJECT CLASSIFICATION. Primary 30E25 93B28 93B05; Secondary 47A50 93C25

KEYWORDS AND PHRASES. Spectral boundary value problems, Right invertible operators, Left invertible operators, Initial boundary operator, Control linear systems, Approximate controllability.

1. DEFINE THE PROBLEM

Let X, E be a complex Banach spaces. This work consists of two parts. In the first part, we develop the following spectral boundary value problems:

$$\begin{cases} Dx = Ax + f \\ \Gamma x = \varphi \end{cases} \tag{1}$$

where $D : \mathcal{D}(D) \subset X \to X$, with $dim\mathcal{N}(D) \neq 0$, be right invertible with a right inverse R, Γ be a boundary operator of D corresponding to $R \in \mathbf{R}_D$ where $\mathbf{R}_D$ is the set of right inverse of D, and A be a linear operator such that $\mathcal{D}(D) \subset \mathcal{D}(A)$. And $f \in X$, $\varphi \in E$ and $\lambda \in \mathbb{C}$ is spectral parameter. We prove the existence and uniqueness the solution of the problem (1).
In the second part of the paper, we develop a theoretical framework for the concepts of controllability. Recall that, in infinite dimensional spaces, exact controllability is not always realized. We give necessary and sufficient conditions for an abstract control linear system to be boundary approximately reachable, boundary exactly controllable and boundary approximately controllable. Finally, by a typical example, we show that the concept and results of the boundary approximate reachability are completely coincide with the approximate reachability of the evolution linear control systems in infinite dimensional spaces.

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ON THE STUDY OF A BOUNDARY VALUE PROBLEM FOR THE BIHARMONIC EQUATION SET IN A IINGULAR DOMAIN

B. CHAOUCHI

Abstract. In this work, we will investigate a boundary value problem for biharmonic equation set in a ingular domain Ω containing a cuspidal point. Existence and maximal regularity results are obtained for the classical solutions by using the fractional powers of linear operators.

2010 Mathematics Subject Classification. 34G10, 34K10, 12H2O, 44A45.

Keywords and phrases. Fractional powers of linear operators; analytic semigroup, Operational differential equation of elliptic type, Cuspidal point

1. STATEMENT OF THE PROBLEM

It is important to note that the study of boundary value problems set in singular domains remains an interesting subject of mathematical analysis. This kind of problems are often encountered in the modeling of many physical phenomena. For this reason, it can be seen that during the last decades numerous authors have been interested in the study of such problems. We can cite for instance [1], [7], [10], [11], [12], [13], [14], [15] and the references therein. Among these problems, a special attention is given to the biharmonic equation. In fact, it is well known that several mathematical models of problems of the plane deformation of the elasticity theory are reduced to the study of the biharmonic equation with some special boundary conditions.

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PERIODICITY OF THE SOLUTIONS OF GENERAL SYSTEM OF RATIONAL DIFFERENCE EQUATIONS RELATED TO FIBONACCI NUMBERS.

IBTISSAM TALHA AND SALIM BADIDJA

Abstract. In this work we deal with the periodicity of solutions of the following system rational of difference equations

$$\begin{cases} x_{n+1} = \frac{y_n(x_{n-3}+y_{n-4})}{y_{n-4}+x_{n-3}-y_n}, \\ \\ y_{n+1} = \frac{x_{n-2}(x_{n-2}+y_{n-3})}{2x_{n-2}+y_{n-3}}, \end{cases}$$

where the initial conditions of the negative index terms

$x_{-3}, x_{-2}, x_{-1}, x_0, y_{-4}, y_{-3}, y_{-2}, y_{-1}, y_0$ are nonzero real numbers and $n = 0, 1, 2, ...$, such that

$$\frac{y_{-3}}{x_{-2}}, \frac{y_{-2}}{x_{-1}}, \frac{y_{-1}}{x_0} \notin \left\{ -\frac{F_{2n+3}}{F_{2n+2}}, n = 0, 1, 2, ... \right\},$$

and

$$\frac{x_{-3} + y_{-4}}{y_0} \notin \{1\} \cup \left\{ \frac{F_{2n}}{F_{2n+2}}, n = 0, 1, 2, ... \right\}.$$

2010 Mathematics Subject Classification. 26A18, 39A05, 39A06.

Keywords and phrases. Periodic solutions, Systems of difference equations, Fibonacci numbers.

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POSITIVE SOLUTION OF A NONLINEAR SINGULAR TWO POINT BOUNDARY VALUE PROBLEM

CHAHIRA ATTIA AND SALIMA MECHROUK

Abstract. In this paper, we study the existence of positive solutions of a nonlinear singular two point Boundary value problem for a class of second order differential equations by using Krasnoselskii's fixed point theorem on cones.

2010 Mathematics Subject Classification. 47H10, 47H11, 34B15.

Keywords and phrases. Cones, singular problem , Krasnoselskii's fixed point theory, existence, positive solution, boundary value problems.

1. Define the problem

This work is concerned with the existence of positive solutions of the following nonlinear second-order singular two point Boundary value problem:

$$\begin{cases} -u''(t) = g(t)\ f(t, u(t)), \quad t \in (0, 1), \\ u'(0) = u(1) = 0 \end{cases} \tag{1}$$

where $g : (0, 1) \to \mathbb{R}^+$ is a measurable function may be singular at $t = 0$ and/or 1 and $f(t, u)$ may also have singularity at $u = 0$. Moreover, The functions g and f satisfy

- (H_1) $\quad 0 < \int_0^1 G(s, s)g(s)ds < +\infty$
- (H_2) $\quad f \in C((0, 1) \times (0, +\infty), \mathbb{R}^+)$

We mean, by a positive solution to problem (1) , a function $u \in C^1([0, 1], \mathbb{R})$ and $u(t_0) > 0$ for some $t_0 > 0$ satisfying all equation in (1). We shall assume some asymptotic properties of f. In particular, assume there exist nonnegative constants in the extended reals, f_0, f_∞, such that

$$f_0 = \lim_{u \to 0^+} \frac{f(t, u)}{u}, \quad f_\infty = \lim_{u \to +\infty} \frac{f(t, u)}{u}$$

We note that the case $(f_0 = 0, f_\infty = \infty)$ corresponds to the superlinear and $(f_0 = \infty, f_\infty = 0)$ corresponds to the sublinear case. we shall also apply the Krasnoselskii's fixed point theorem on cones on which there exist positive solutions of the BVP (1).

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PRODUCTS AND COMMUTATIVITY OF DUAL TOEPLITZ OPERATORS ON THE HILBERTIAN HARDY SPACE OF THE POLYDISK

LAKHDAR BENAISSA

Abstract. In this paper, we study the commutativity and products of dual Toeplitz operators on the Hardy space of the polydisk, we obtain similar conditions of BrownHalmos Theorem for Hardy-Dual Toeplitz operators, and establish their main algebraic properties using an auxiliary transformation of operators

2010 Mathematics Subject Classification. 47B35, 47B47.

Keywords and phrases. Brown-Halmos, dual Toeplitz operator, Hankel operator, Hardy space.

1. Define the problem

We introduce dual Toeplitz operators on the orthogonal complement of the Hardy space of the polydisk and establish their main algebraic properties using an auxiliary transformation of operators. For a detailed account on this topic we refer to [10, 5]. This mysterious transformation gives rise to an interesting characterization of dual Toeplitz operators in terms of operator equations that is closely related to the intertwining relations. Furthermore, we are able to characterize commuting dual Toeplitz operators as well as normal ones. Moreover, we investigate products of dual Toeplitz operators. More precisely, we establish Brown-Halmos type theorems and exploit them to characterize the zero divisors among dual Toeplitz operators as well as symbols giving rise to isometric, idempotent and unitary dual Toeplitz operators.

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Polynomial stability of a circular arch problem with boundary dissipation conditions

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Abstract
In this work, we study the asymptotic stability of a **circular arch problem** with a boundary control of fractional derivative type in the sense of Caputo.

Keywords :

- Circular arch problem ,
- Boundary controls,
- Asymptotic stability.

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Positive solutions for second order boundary value problems with dependence on the first order derivative

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Abstract:

In this Work, We study the existence of positive solutions for nonlocal boundary value problems for functional differential equations

$$\begin{aligned}&u''(t) + f(t, u_t, u'(t)) = 0, \quad 0 \leq t \leq 1,\\&u(t) = \phi(t), \quad -\tau \leq t \leq 0,\\&u(1) = \alpha u(\eta) + \beta u'(\eta)\end{aligned}$$

where $\phi \in C$, $f : [0, 1] \times C \times \mathbb{R} \to \mathbb{R}$ is continuos functions and $\eta \in (0, 1)$,

keywords: Positive solution, functional differential equation, nonlocal boundary value problem, alternative of Leray-Schauder fixed point theorem.

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Reconstruction of an unknown time-dependent source parameter in a time-fractional Sobolev-type problem from overdetermination condition.

Abdeldjalil Chattouh, Khaled Saoudi.

Let $\Omega \subset \mathbb{R}^d, d \geq 1$ is a bounded domain with a Lipschitz boundary Γ and $T > 0$ is a final time. Consider the following time-fractional Sobolev-type equation

$$\partial_t^\alpha u(x,t) - \partial_t^\alpha \Delta u(x,t) - \Delta u(x,t) = h(t) f(x,t), \quad x \in \Omega, t \in (0,T). \tag{1}$$

where ∂_t^α stands for Caputo fractional derivative of order α in the time variable given by

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \partial_t u(x,\tau) \mathrm{d}\tau.$$

Note that the equation (1) is a classical diffusion and wave equation for $\alpha = 1$ and $\beta = 2$, respectively. We associate to the equation (1) the following initial and boundary conditions

$$\begin{aligned} u(x,0) = u_0(x), \quad x \in \Omega, \\ -\nabla u(x,t) \cdot \nu = g(x,t), \quad x \in \Gamma, \quad t \in (0,T). \end{aligned} \tag{2}$$

where the initial data u_0 and the source term g are given smooth functions, and the symbol ν stands for the outer normal vector assigned to the boundary Γ.
Define the Riemann–Liouville kernel as

$$g_{1-\alpha}(t) = \frac{t^{1-\alpha}}{\Gamma(1-\alpha)}, \quad t > 0, \quad 0 < \alpha < 1.$$

and the convolution on the positive half-line, i.e.

$$(k * v)(t) = \int_0^t k(t-\tau) v(\tau) \mathrm{d}\tau.$$

Thus, the equation (1) can be written in the following equivalent form

$$(g_{1-\alpha} * \partial_t u(x))(t) - (g_{1-\alpha} * \partial_t \Delta u(x))(t) - \Delta u(x,t) = h(t) f(x,t), \quad x \in \Omega, t \in (0,T). \tag{3}$$

The Inverse source problem studied in this contribution consists of finding a couple $(u(x,t), h(t))$ satisfying (1), (2) and the following overdetermination condition:

$$\int_\Omega u(x,t) \omega(x) \mathrm{d}x = m(t), \quad t \in [0,T]. \tag{4}$$

where ω is a space-dependent function. Usually ω is chosen to be a function with compact support in Ω, and then this type of measurement represents the weighted average of u on a subdomain of Ω.

Multiplying (**??**) by the function ω, and integrating over the domain Ω, applying the Green theorem and using (4), we obtain

$$(g_{1-\alpha} * m')(t) + (\nabla u(t), \nabla \omega) = h(t)(f, \omega) - (g(t), \omega)_\Gamma, \quad 0 < t \leq T. \tag{5}$$

Assume that $(f, w) \neq 0$, than

$$h(t) = \frac{(g_{1-\alpha} * m')(t) + (\nabla u(t), \nabla \omega) + (g(t), \omega)_\Gamma}{(f, \omega)}, \quad 0 < t \leq T. \tag{6}$$

Similarly multiplying (1) by a test function $\phi \in H^1(\Omega)$ and using Green formula, we obtain the variational formulation of (3) and (2), which reads as

$$((g_{1-\alpha} * \partial_t u)(t), \phi) + (\nabla u(t), \nabla \phi) = h(t)(f, \phi) - (g(t), \phi)_\Gamma, \quad 0 < t \leq T. \tag{7}$$

for any $\phi \in H^1(\Omega)$ and $u(0) = u0$. The relations (5) and (7) represent the variational formulation of the inverse problem (1), (2) and (4).

Direct and inverse problems of parabolic type containing a source parameter and/or an integral over (a subset of) the spatial domain of a function of the unknown solution arise in the to modelling various physical phenomena. The integral may appear in the boundary conditions and/or the governing partial differential equation itself. These problems have many applications in fields of science and engineering, e.g. in thermoelasticity and in fluid flow, in heat transfer processes, in control theory, in chemical diffusion and in vibration problems.

Identification of an unknown source is an activate and interesting topic in the theory of inverse problems. There are many papers devoted to the study of inverse problems for parabolic and hyperbolic equations. The case when the source depends only on the space variable, we refer to [1,2,3], and for the solely time-dependent source reader can see [4,5,6]. An unknown time-dependent source function $h(t)$ appears in paper [7], that deals with an inverse problems for time-fractional wave equation along with mixed boundary conditions. In that article, $p(t)$ is recovered from the boundary measurement overdetermination (4).

The main objective of this work is to establish the uniqueness and global existence of the weak solution of the inverse problem (2)-(1). Following the same idea as in [] and [**?**] we employ the Rothe method, which is a very powerful tool approaching problem complexly. We prove the existence of solution in a constructive way, which enables us also to propose an algorithm to compute an approximate solution of the inverse problem.

Our main result in this contribution lies in the following theorem

Theorem Let $f \in L^2(\Omega)$, $u_0, \omega \in H^1(\Omega)$, $\int_\Omega f\omega \neq 0, m \in \mathcal{C}^2([0,T])$. Suppose that

$$(\nabla u_0, \nabla \phi) = h_0(f, \phi) - (g_0, \phi)_\Gamma, \forall \phi H^1(\Omega),$$

Then there exists a unique solution (u, h) to the (5) and (7) obeying that $u \in \mathcal{C}([0,T]; H^1(\Omega))$ with $u_t \in L^\infty((0,T); L^2(\Omega)) \cap \mathcal{C}([0,T]; L^2(\Omega))$ and $h \in \mathcal{C}([0,T])$.

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STABILITY WITH RESPECT TO PART OF THE VARIABLES OF NONLINEAR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

ABDELLATIF BEN MAKHLOUF

ABSTRACT. In this work, the stability with respect to part of the variables of nonlinear Caputo fractional differential equations is studied. A sufficient conditions of stability, uniform stability, Mittag Leffler stability and asymptotic uniform stability of this type are obtained within the method of Lyapunov-like function.

2010 MATHEMATICS SUBJECT CLASSIFICATION.26A33, 65L20.

KEYWORDS AND PHRASES. Fractional order system, stability analysis, Mittag-Leffler function.

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STRONG SOLUTION FOR HIGH-ORDER CAPUTO TIME FRACTIONAL PROBLEM WITH BOUNDARY INTEGRAL CONDITIONS

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Abstract. The aim of this paper is to work out the solvability of a class of Caputo time fractional problems with boundary integral conditions. A generalized formula of integration is demonstrated and applied to establish the a priori estimate of the solution, then we prove the existence which is based on the range density of the operator associated with the problem.

2010 Mathematics Subject Classification. 35R11, 35D35.

Keywords and phrases.Time fractional problem, a priori estimate, boundary integral conditions.

1. Define the problem

Let Q be a rectangle defined by $Q = (0,1) \times (0,T)$ and considering the fractional equation

$$\partial_{0t}^{\alpha} u + (-1)^m \frac{\partial^m}{\partial x^m}\left(a(x,t) \frac{\partial^m u}{\partial x^m} \right) = f(x,t) \tag{1}$$

where $m \geq 1$ and ∂_{0t}^{α} denotes the Caputo time fractional derivative of order $0 < \alpha < 1$ with lower bound 0, subject to the initial condition

$$u(x,0) = \varphi(x), \qquad x \in (0,1) \tag{2}$$

and the boundary integral conditions

$$\int_0^1 x^k u(x,t)\, dx = 0, \qquad k = \overline{0, 2m-1}. \tag{3}$$

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SOLUTIONS FORMULAS FOR SOME GENERAL SYSTEMS OF DIFFERENCE EQUATIONS

Y. AKROUR, M. KARA, N. TOUAFEK, AND Y. YAZLIK

Abstract. In this work, we give explicit formulas of the solutions of the two general systems of non-linear difference equations

$$x_{n+1} = f^{-1}\left(ag(y_n) + bf(x_{n-1}) + cg(y_{n-2}) + df(x_{n-3})\right),$$

$$y_{n+1} = g^{-1}\left(af(x_n) + bg(y_{n-1}) + cf(x_{n-2}) + dg(y_{n-3})\right),$$

and

$$x_{n+1} = f^{-1}\left(a + \frac{b}{g(y_n)} + \frac{c}{g(y_n)f(x_{n-1})} + \frac{d}{g(y_n)f(x_{n-1})g(y_{n-2})}\right),$$

$$y_{n+1} = g^{-1}\left(a + \frac{b}{f(x_n)} + \frac{c}{f(x_n)g(y_{n-1})} + \frac{d}{f(x_n)g(y_{n-1})f(x_{n-2})}\right),$$

where $n \in \mathbb{N}_0$, $f, g : D \longrightarrow \mathbb{R}$ are a "$1-1$" continuous functions on $D \subseteq \mathbb{R}$, the initial values x_{-i}, y_{-i}, $i = 0, 1, 2, 3$ are arbitrary real numbers in D and the parameters a, b, c and d are arbitrary real numbers. Our results considerably extend some existing results in the literature.

2010 Mathematics Subject Classification. 39A10.

Keywords and phrases. Systems of difference equations, form of solutions, stability of equilibrium points.

1. Define the problem

Difference equations are used to describes real discrete models in various branches of modern sciences such as, biology, economy, control theory. This explain why a big number of papers is devoted to this subject, see for example ([1] - [21]). It is clear that if we want to understand our models, we need to know the behavior of the solutions of the equations of the models, and this fact will be possible if we can solve in closed form these equations. One can find in the literature a lot of works on difference equations where explicit formulas of the solutions are given, see for instance [1], [2], [5], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [20], [21]. Such type of difference equations and systems is called solvable difference equations. In the present work, we continue our interest in solvable difference equations, more precisely, we will solve the following two general systems of difference equations

$$x_{n+1} = f^{-1}\left(ag(y_n) + bf(x_{n-1}) + cg(y_{n-2}) + df(x_{n-3})\right),$$

$$y_{n+1} = g^{-1}\left(af(x_n) + bg(y_{n-1}) + cf(x_{n-2}) + dg(y_{n-3})\right),$$

and

$$x_{n+1}=f^{-1}\left(a+\frac{b}{g(y_n)}+\frac{c}{g(y_n)f(x_{n-1})}+\frac{d}{g(y_n)f(x_{n-1})g(y_{n-2})}\right),$$

$$y_{n+1}=g^{-1}\left(a+\frac{b}{f(x_n)}+\frac{c}{f(x_n)g(y_{n-1})}+\frac{d}{f(x_n)g(y_{n-1})f(x_{n-2})}\right),$$

where $n\in\mathbb{N}_0$, $f,g:D\longrightarrow\mathbb{R}$ are one to one ("$1-1$") continuous functions on $D\subseteq\mathbb{R}$, the initial values x_{-i}, y_{-i}, $i=0,1,2,3$ are arbitrary real numbers in D and the parameters a, b, c and d are arbitrary real numbers.

In our study, we are inspired and motivated by the ideas, the equations and the systems of some recent published papers. The papers, [1], [2] and especially [15] are our main motivation in the present work. The obtained results considerably generalize some existing results in the literature, see [1], [2], [3], [4], [5], [11], [12], [13], [14], [15], [16], [17], [18], [20].

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SOLUTIONS OF THE OPERATOR EQUATIONS $T^n = T^*T$

SOUHEYB DEHIMI

Abstract. In this paper, we study equations of the type $T^*T = T^n$ where T is a linear operator (not necessary bounded) and $n \in \mathbb{N}$ and see when they yield $T = T^*$.

2010 Mathematics Subject Classification. Primary 47A62. Secondary 47B20, 47B25.

Keywords and phrases. Operators equation, Self-adjoint operators. Quasinormal operators. Spectrum.

1. Introduction

It was proved in [10] that if $T^*T = T^2$ then T must be self-adjoint ($T = T^*$), where $T \in B(H)$ and H is a finite dimensional space. Then, the authors in [7] obtained positive results in both the finite and the infinite dimensional settings. In this paper, we deal with more general equations of the type

$$T^*T = T^n, \quad n \in \mathbb{N},$$

where T is closed linear operator. We also reprove some known results in [7] by using the proprieties of posinormal operators.

2. Main results

Definition 2.1. *Let $T \in B(H)$. If $T^*T = T^n$ for some $n \in \mathbb{N}$ such that $n \geq 3$, then T is called a generalized projection.*

This class of operators was first defined by the others in [2]. Note that, $T^*T = T^n$ does not always gives the self-adjointness of $T \in B(H)$ even when $\dim H < \infty$. This new class of operators lies therefore just between orthogonal projections and normal operators.

The next theorem was proved in [2], which might considered as a characterization of the results obtained in [10].

Theorem 2.2. *Let H be a complex Hilbert space and let $T \in B(H)$ be a bounded operator and let $n \in \mathbb{N}, n \geq 2$. Then T is a solution of the equality*

$$T^n = T^*T \tag{1}$$

if and only if

- $T = T^*$ *(if $n = 2$),*
- *there is a family $P_1, \ldots, P_n \in B(H)$ of orthogonal projections such that $P_jP_k = 0, (j \neq k)$ such that*

$$T = \sum_{k=1}^{n} e^{\frac{2k\pi i}{n}} P_k \tag{2}$$

(if $n \geq 3$). In this case, we also have $\|T\| = 1$ (when $A \neq 0$).

This new class of operators lies therefore just between orthogonal projections and normal operators.

As an immediate consequence, we have:

Proposition 2.3. *If $B, C \in B(H)$ are such that $C^*C = BC$ and $B^*B = CB$, then $B = C^*$.*

Proof. Let $T \in B(H \oplus H)$ be defined as $T = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. Then

$$T^*T = \begin{pmatrix} C^*C & 0 \\ 0 & B^*B \end{pmatrix} \text{ and } T^2 = \begin{pmatrix} BC & 0 \\ 0 & CB \end{pmatrix}.$$

By hypothesis, we ought to have $T^*T = T^2$, whereby T becomes self-adjoint, in which case, $B = C^*$, as wished. □

Proposition 2.4. *Let $T \in B(H)$ be such that $T^*T^2 = T^*TT^*$. Then T is self-adjoint.*

Another consequence is the following:

Proposition 2.5. *Let $T \in B(H)$ be satisfying*

$$T^*T = T^{*2}T^2 = T^3.$$

Then there exist three orthogonal projections, P_0, P_1, $P_2 \in B(H)$ which are pairwise orthogonal such that

$$T = P_0 + e^{\frac{2\pi i}{3}}P_1 + e^{\frac{4\pi i}{3}}P_2. \tag{3}$$

Proof. Let $T \in B(H)$ and define $B \in B(H \oplus H)$ by:

$$B = \begin{pmatrix} 0 & T \\ T^2 & 0 \end{pmatrix}.$$

Then $B^2 = \begin{pmatrix} T^3 & 0 \\ 0 & T^3 \end{pmatrix}$. Since $B^*B = \begin{pmatrix} T^{*2}T^2 & 0 \\ 0 & T^*T \end{pmatrix}$, by hypothesis we must therefore have $B^*B = B^2$. Hence, B is self-adjoint by Theorem 2.2. This just means that $T = T^{*2}$. Consequently, T is obviously normal and

$$\varphi(z) = z - \overline{z}^2,\ z \in \mathbb{C}$$

vanishes on $\sigma(T)$. From that it is readily seen that if $\lambda \in \sigma(T)$ then either $\lambda = 0$ or λ is a solution of $\lambda^3 = 1$. Whence, we conclude that

$$\sigma(T) \subseteq \{0\} \cup \{e^{\frac{2k\pi i}{3}},\ k = 0, 1, 2\}.$$

From the spectral theorem it follows that T can be written as (3) for some orthogonal projections P_0, P_1, P_2 with pairwise orthogonal ranges. The proof is complete. □

Now, we deal with the equation $T^*T = T^n$ for a closed and densely defined T.

In fact, If T is a linear operator, then $T^*T = T^2$ does not necessarily give $T = T^*$. The most trivial example is to consider a densely defined and unclosed operator T (hence such T cannot be self-adjoint) such that

$$D(T^2) = D(T^*T) = \{0\}$$

. Then $T^*T = T^2$ is trivially satisfied.

Theorem 2.6. *Let H be a complex Hilbert space and let T be a closed and densely defined (unbounded) operator verifying $T^*T = T^2$. Then T is self-adjoint on its domain $D(T) \subset H$.*

Proof. Plainly,

$$T^*T = T^2 \Longrightarrow TT^*T = T^3 \Longrightarrow TT^*T = T^2T \Longrightarrow TT^*T = T^*TT,$$

showing the quasinormality of T (as defined in [3], say). By consulting [4] and [6], we know that quasinormal operators are hyponormal. That is, T is hyponormal.

According to the proof of Theorem 8 in [1], closed hyponormal operators having a real spectrum are automatically self-adjoint. Once that's known and in order that T be self-adjoint, it suffices therefore to show the realness of its spectrum given that T is already closed.

So, let $\lambda \in \sigma(T)$. Since T is closed, we have by invoking a spectral mapping theorem (e.g. Theorem 2.15 in [5]) that $\lambda^2 \geq 0$ for T^*T is self-adjoint and positive. Now, this forces λ to be real. Accordingly, $\sigma(T) \subset \mathbb{R}$, as needed. □

As a consequences of the previous theorem, we have:

Proposition 2.7. *Let B, C be two densely defined and closed operators obeying $C^*C = BC$ and $B^*B = CB$. Then $B = C^*$.*

Proposition 2.8. *Let T be a closed and densely defined (unbounded) operator such that $T^*T = -T^2$. Then T is skew-adjoint.*

Finally, we show the impossibility of the equations $T^*T = T^n$ (with $n \geq 3$) for unbounded closed operators.

Theorem 2.9. *Let T be a closed and densely defined operator with a domain $D(A) \subset H$ and let $n \in \mathbb{N}$ be such that $n \geq 3$. If $T^*T = T^n$, then $T \in B(H)$ (and so T can be written in the form (2)).*

Proof. Let T be a closed and densely defined operator which obeys $T^*T = T^n$ where $n \geq 3$. Then (as in the bounded case)

$$T^*T = T^n \Longrightarrow TT^*T = T^{n+1} \Longrightarrow TT^*T = T^nT \Longrightarrow TT^*T = T^*TT,$$

showing the quasinormality of T. It then follows that T is hyponormal and so $D(T) \subset D(T^*)$. Hence

$$D\left(T^2\right) \subseteq D\left(T^*T\right) = D\left(T^n\right)$$

or merely

$$D\left(T^2\right) = D\left(T^n\right).$$

Also

$$D\left(T^3\right) \subseteq D\left(T^*TT\right) = D\left(T^{n+1}\right)$$

so that

$$D\left(T^2\right) = D\left(T^{n+1}\right).$$

Now, since T is closed, it follows that T^2 is closed as it is already quasinormal (see e.g. Proposition 5.2 in [8]). Also, the quasinormality of T yields that of T^2 (by Corollary 3.8 in [3], say) and so T^2 is hyponormal. Therefore,

$$D\left(T^2\right) \subseteq D[(T^2)^*]$$

and

$$D\left(T^2\right) = D\left(T^4\right).$$

In the end, according to Corollary 2.2 in [9], it follows that T^2 is everywhere bounded on H. Hence $D(T) = H$ and so the Closed Graph Theorem intervenes now to make $T \in B(H)$. □

Declaration. This work is inspired by the original paper "On the operator equations $A^n = A^*A$".

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SOME FIXED POINT RESULTS FOR KHAN MAPPINGS

SAMI ATAILIA, NAJEH REDJEL, AND ABDELKADER DEHICI

ABSTRACT. We present some fixed point results for a class of mappings called Khan mappings which satisfy certain rational inequality. Furthermore, we establish the link that connects quasi-normal-structure and the fixed point property for this class when they are defined on weakly compact convex subsets of Banach spaces.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 47H10, 54H25.

KEYWORDS AND PHRASES. Khan mapping, fixed point, iterative process, Picard sequence, quasi-nonexpansive mapping, quasi-normal structure.

1. DEFINE THE PROBLEM

We focus our study on the existence of fixed points for Khan self-mappings (involving rational expression) which are defined in complete metric spaces.

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SOME COUPLE FIXED POINT THEOREMS IN METRIC SPACE ENDOWED WITH GRAPH

A. BOUDAOUI AND K. MEBARKI

Abstract. In this presentation, we will talk about the sufficient conditions for the existence of couple fixed point for such contractive mappings in mertic space endowed with a directed graph. Our results represent a generalizations of the recent couple fixed point theorems given by Vasile Berinde [2]. We apply the proven couple fixed point results on the existence and the uniqueness of a continuous solution for a system of fractional differential equation.

2010 Mathematics Subject Classification. 47H10, 54H25, 54E35, 34A08

Keywords and phrases. couple fixed point, metric space, directed graph, mixed G-monotone, Fractional differential equations.

1. Define the problem

The study of coupled fixed point theorems remain a well motivated area of research in fixed point theory due to their applications in a wide variety of problems. Bhaskar and Lakshmikantham [3], Vasile Berinde [2], Van Luong and Thuan [12] presented some new results for coupled fixed point in partially ordered metric space and used them to prove the existence and uniqueness of some differential equations.

In 2008, Jachymski [8] gave a more general unified version of the results obtained in metric spaces endowed with a partial order by considering graphs instead of a partial order. In this direction, Bojor [4], Boonsri and Saejung[5], Chifu and Petruşel [6] presente some fixed point theorems in metric space endowed with graph.

Very recently, Alfuraidan and Khamsi [1], Chifu and Petrusel [7] have developed have developed some coupled fixed point results in metric space endowed with a directed graph.

Following the same line, in this manuscript we give a generalization of Vasile Berinde's theorem in metric space with graph. As an application, we prove the existence and the uniqueness of a continuous solution for a system of fractional differential equation by using the results obtained.

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STABILITY OF FIRST ORDER DELAY INTEGRO-DYNAMIC EQUATIONS ON TIME SCALES

KAMEL ALI KHELIL AND ABDELOUAHEB ARDJOUNI

Abstract. The main purpose of the present work was to establish some basic theory of time scale calculus which is an efficient mathematical theory that unifies discrete and continuous calculus. However, we apply the contraction mapping theorem to obtain asymptotic stability results about the zero solution for first order delay integro-dynamic equation. An asymptotic stability theorem with a necessary and sufficient condition is proved. In addition, the case of the equation with several delays is studied.

2010 Mathematics Subject Classification. 34K20, 34N05, 45J05.

Keywords and phrases. Delay dynamic equations, fixed point, stability, time scales.

1. The main results

For the convenience of the reader, let us recall the definition of asymptotic stability. For each t_0, we define

$$m(t_0) = \min(\inf\{s - r(s) : s \geq t_0\}, \inf\{s - h(s) : s \geq t_0\})$$

and denote $C_{rd}(t_0)$ the space of rd-continuous functions on $[m(t_0), t_0]$ with the supremum norm $\|.\|_{t_0}$.

For each $(t_0, \phi) \in \mathbb{T} \times C_{rd}(t_0)$, denoted by $x(t) = x(t, t_0, \phi)$ the unique solution of the equation

$$x^{\Delta}(t) + \int_{t-r(t)}^{t} a(t,s)x(s)\Delta s + b(t)x(t - h(t)) = 0 \tag{1}$$

and $x(t) = \phi(t), t \in [m(t_0), t_0]$.

The zero solution of Eq (1) is called

(i) stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x(t, t_0, \phi)| < \varepsilon$ for all $t \geq t_0$ if $\|\phi\|_{t_0} < \delta$.

(ii) asymptotically stable if it is stable and $\lim_{t\to\infty} |x(t, t_0, \phi)| = 0$.

Theorem 1.1. *Suppose that the following two conditions hold:*

$$\liminf_{t\to\infty} \int_0^t \frac{1}{\mu(\tau)} \log(1 + \mu(\tau)A(\tau))\Delta\tau > -\infty, \tag{2}$$

$$\sup_{t\geq 0} \int_0^t \omega(s)e_{\ominus A}(t,s)\Delta s = \alpha < 1, \tag{3}$$

where

$$A(\tau) = \int_{\tau - r(\tau)}^{\tau} a(\tau, s)\Delta s + b(\tau), \quad A(\tau) \in \mathcal{R}^+$$

and

$$\omega(s) = \int_{s-r(s)}^{s} |a(s,w)| \int_{w}^{\sigma(s)} \left(\int_{u-r(u)}^{u} |a(u,v)| \,\Delta v + |b(u)| \right) \Delta u \Delta w$$
$$+ |b(s)| \int_{s-h(s)}^{\sigma(s)} \left(\int_{u-r(u)}^{u} |a(u,v)| \,\Delta v + |b(u)| \right) \Delta u.$$

Then the zero solution of (1) is asymptotically stable if and only if

$$\int_0^t \frac{1}{\mu(\tau)} \log(1 + \mu(\tau)A(\tau))\Delta\tau \to \infty \text{ as } t \to \infty. \tag{4}$$

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STABILITY OF THE SCHRÖDINGER EQUATION WITH A TIME VARYING DELAY TERM IN THE BOUNDARY FEEDBACK

WASSILA GHECHAM, SALAH-EDDINE REBIAI, AND FATIMA ZOHRA SIDI ALI

Abstract. In recent years, stability analysis of PDE systems with delay has received a considerable amount of attention; see for example [1], [7] and the references therein. In [6], Nicaise et al used Lyapunov-based technique to establish sufficient conditions to guarantee the exponential stability of the solution of the one-dimensional wave equation with boundary time-varying delays. This result was extended to general space dimension in [4]. Our aim in this paper is to study the stability problem for the Schrödinger equation with a time-varying delay term in the boundary feedback. This problem was considered in [2] and [3] in the absence of delay and in [5] for the case of constant delay. Under suitable assumptions, we prove exponential stability of the solution. This result is obtained by introducing a suitable energy function and by constructing a suitable Lyapunov functional.

2010 Mathematics Subject Classification. 93D15; 35J10.

Keywords and phrases. Schrödinger equation, time-varying delay, stabilization, boundary feedback.

1. Define the problem

Let Ω be an open bounded domain of $\mathbb{R}^n$ with boundary Γ of class C^2 which consists of two non-empty parts Γ_1 and Γ_2 such that, $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$. In Ω, we consider a Schrödinger equation with a time varying delay term in the boundary feedback:

(1)
$$\begin{cases} u_t(x,t) - i\Delta u(x,t) = 0 & \text{in } \Omega \times (0;+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = i\alpha_1 u(x,t) + i\alpha_2 u(x,t-\tau(t)) & \text{on } \Gamma_2 \times (0,+\infty), \\ u(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{on } \Gamma_2 \times (0,\tau(0)), \end{cases}$$

where

- u_0 and f_0 are the initial data which belong to suitable spaces.
- $\frac{\partial}{\partial \nu}$ is the normal derivative.
- $\tau(t)$ is the time varying delay.
- α_1 and α_2 are positive constants.

The main purpose of this work is to prove exponential stability of the system (1).

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TWO-DIMENSIONAL HARDY INTEGRAL INEQUALITIES WITH PRODUCT TYPE WEIGHTS

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Abstract. In this work, we give some new two-dimensional weighted Hardy integral inequalities by using weighted mean operator $H_\phi f$, where f nonnegative integrable function with two variables on $\Omega = (0, +\infty) \times (0, +\infty)$ and ϕ is a weight function.

2010 Mathematics Subject Classification. 26D10, 26D15.

Keywords and phrases. Hölder's inequality, weight function.

1. Introduction

The inequality

$$\int_0^\infty x^{-m} F^q(x)dx \leq \left(\frac{q}{m-1}\right)^q \int_0^\infty x^{-m}(xf(x))^q(x)dx, \tag{1}$$

where $F(x) = \displaystyle\int_0^x f(t)dt$, $\quad m > 1$, known as the generalization of Hardy's inequality, is satisfied for all functions f non-negative and measurable on $(0, \infty)$ with $q > 1$. The constant is the best possible. The aim of this presentation is to give a new two-dimensional weighted Hardy integral inequalities by using some elementary methods of analysis and the weighted Hardy operator $H =: H_\phi f$.

2. Main result

Let $0 < a < b < +\infty$ and $0 < c < d < +\infty$. We will assume that the function f is nonnegative integrable on $\Omega = (0, +\infty) \times (0, +\infty)$ and the integrals throughout are assumed to exist and are finite.

Theorem 2.1. *Suppose f nonnegative integrable on Ω and $q > 1$, $m > 1$. Let*

$$H(x, y) = \frac{1}{\Phi(x)\Phi(y)} \int_a^x \int_c^y \phi(t)\phi(s)f(t, s)\, ds\, dt,$$

and

$$\Phi(z) = \int_0^z \phi(s)ds.$$

If $\lambda \geq \dfrac{m-1}{q+m-1}$, then

$$\begin{aligned} &\int_a^b \int_c^d \frac{\phi(x)\phi(y)}{\Phi^m(x)\Phi^m(y)} H^q(x, y)dydx \\ &\leq \left(\frac{\lambda q}{m-1}\right)^{2q} \int_a^b \int_c^d \frac{\phi(x)\phi(y)}{\Phi^m(x)\Phi^m(y)} f^q(x, y)dy\, dx. \end{aligned} \tag{2}$$

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3. Applications

Two-Dimensional Weighted Hardy Integral Inequalities
If we put $\phi(x) = 1$ in Theorem 2.1, we have the following corollary.

Corollary 3.1. *Suppose $q > 1$, $m > 1$ and f be nonnegative integrable function on Φ. Let*

$$F(x,y) = \frac{1}{xy}\int_a^x \int_c^y f(t,s)\,ds\,dt.$$

If $\lambda \geq \dfrac{m-1}{q+m-1}$, then

(3)
$$\int_a^b \int_c^d (xy)^{-m} F^q(x,y)dydx \leq \left(\frac{\lambda q}{m-1}\right)^{2q} \int_a^b \int_c^d (xy)^{-m} f^q(x,y)dydx.$$

Bilinear Hardy Inequality
Suppose $f(x,y) = f_1(x).f_2(y)$ where $f_1,\ f_2$ are nonnegative integrable functions on $(0,\infty)$, from the Corollary 3.1 we obtain a special bilinear case.

Corollary 3.2. *Let $q > 1$, $m > 1$ and*

$$F(x,y) = \left(\frac{1}{x}\int_a^x f_1(t)\,dt\right)\left(\frac{1}{y}\int_c^y f_2(r)\,dr\right).$$

If $\lambda \geq \dfrac{m-1}{q+m-1}$, then

$$\begin{aligned} \int_a^b \int_c^d (xy)^{-m} F^p(x,y)dydx \quad &\leq \left(\frac{\lambda q}{m-1}\right)^{2q} \left(\int_a^b x^{-m} f_1^q(x)dx\right) \\ &\times \left(\int_c^d y^{-m} f_2^q(y)dy\right). \end{aligned} \tag{4}$$

One-Dimensional Analogue Of The Initial Inequality (1)
If we put $f_1 = f_2,\ a = c,\ b = d$ in the Corollary 3.2, we obtain

Corollary 3.3. *Suppose $q > 1$, $m > 1$ and f nonnegative integrable on $(0,\infty)$. Let*

$$F(x) = \frac{1}{x}\int_a^x f(t)\,dt.$$

If $\lambda \geq \dfrac{m-1}{q+m-1}$, then

$$\int_a^b x^{-m} F^q(x)dx \leq \left(\frac{\lambda q}{m-1}\right)^q \int_a^b x^{-m} f^q(x)dx. \tag{5}$$

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THE EXISTENCE OF TWO SOLUTIONS FOR STEKLOV PROBLEM INVOLVING THE $p(x)$-LAPLACIAN

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Abstract. In this work, By using variational methods and mountain pass Lemma combined with Ekeland variational principle, and for some hypothesis, we prove the existence of two nontrivial weak solutions to a class of $p(x)$-Laplacian problems.

2010 Mathematics Subject Classification. 35J48, 35J60, 35J66.

Keywords and phrases. $p(x)$-Laplacian, Ekeland principle, Variational methods.

1. Define the problem

The goal of this paper, is to study the following Steklov boundary value problem

(1)
$$\begin{cases} -\operatorname{div}\left(a(x)\,|\nabla u|^{p(x)-2}\,\nabla u\right) + |u|^{p(x)-2}u = f(x,u) + \lambda\,|u|^{\gamma(x)-2}\,u \text{ in } \Omega, \\ \\ a(x)\,|\nabla u|^{p(x)-2}\,\frac{\partial u}{\partial v} + b(x)|u|^{q(x)-2}u = h(x,u) \text{ on } \partial\Omega. \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N (N \geq 2)$ with Lipschitz boundary $\partial\Omega, \frac{\partial}{\partial v}$is the outer unit normal derivative, $p(x), \gamma(x)$ are continuous functions on $\overline{\Omega}$ such that $1 < p^- = \inf_{\overline{\Omega}} p(x) \leq p(x) \leq \sup_{\overline{\Omega}} p(x) = p^+$, we also denote γ^-, γ^+ for any $\gamma(x) \in C(\overline{\Omega})$ and q^-, q^+ for any $q(x) \in C(\partial\Omega), p(x) \neq \gamma(x) \neq q(y)$ for any $x \in \overline{\Omega}, y \in \partial\Omega, a$ and b are continuous functions such that

$$a_1 \leq a(x) \leq a_2, \text{ and } b_1 \leq b(x) \leq b_2,$$

where a_1, a_2, b_1 and b_2 are positive constants, λ is a positive parameter, $(-\Delta)_{p(x)}u = -div(|\nabla u|^{p(x)-2}\nabla u)$ denotes the $p(x)$-Laplacian, $f : \Omega \times \mathbb{R} \to \mathbb{R}, h : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are caratheodory functions satisfying some conditions. More precisely, we assume the following hypothesis.
(A_1)There exist $M_1, M_2 > 0, \alpha \in C(\overline{\Omega})$ and $\beta \in C(\partial\Omega)$, such that

$$|f(x,u)| \leq M_1\left(1 + |u|^{\alpha(x)-1}\right), \text{ for all } (x,u) \in \Omega \times \mathbb{R},$$

$$|h(x,u)| \leq M_2\left(1 + |u|^{\beta(x)-1}\right), \text{ for all } (x,u) \in \partial\Omega \times \mathbb{R},$$

where

$$1 < \alpha(x) < p^*(x), x \in \Omega \text{ and } 1 < \beta(x) < p_*(x), q(x) < p_*(x), x \in \partial\Omega.$$

(A_2) $f(x,u) = o\left(|u|^{p^+-1}\right)$ as $u \to 0$ for all $x \in \Omega$ and $g(x,u) = o\left(|u|^{p^+-1}\right)$ as $u \to 0$ for all $x \in \partial\Omega$.
(A_3) There exist $R_1, R_2 > 0, \theta_1, \theta_2 > p^+$ such that

$$0 < \theta_1 F(x,u) \leq f(x,u)u,\ |u| \geq R_1,\ \text{for all } x \in \Omega,$$
$$0 < \theta_2 H(x,u) \leq h(x,u)u,\ |u| \geq R_2,\ \text{for all } x \in \partial\Omega,$$

where $F(x,t) = \int_0^t f(x,s)ds, H(x,t) = \int_0^t h(x,s)ds$.
(A_4) There exist $1 < t_1 < p^-$, such that

$$\liminf_{u\to 0} \frac{F(x,u) + \frac{\lambda}{\gamma(x)} |u|^{\gamma(x)}}{|u|^{t_1}} > 0,\ \text{for all } x \in \Omega.$$

(A_5) There exist $1 < t_2 < q^-$, such that,

$$\liminf_{u\to 0} \frac{H(x,u)}{|u|^{t_2}} > 0,\ \text{for all } x \in \partial\Omega.$$

where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)},\ \text{if } p(x) < N, \\ \infty,\ \text{if } p(x) \geq N. \end{cases}, p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)},\ \text{if } p(x) < N, \\ \infty,\ \text{if } p(x) \geq N. \end{cases},$$

we have our main result

Theorem 1.1. *If* $\min\left(\alpha^-, \beta^-\right) > p^+, \min(\theta_1, \theta_2) > q^+$ *and* $(A_1) - (A_5)$ *are satisfied. Then, there exists* $\lambda_0 > 0$ *such that for every* $\lambda \in (0, \lambda_0)$, *problem* (1) *has at least two non trivial solutions.*

To prove Theorem 1.1, we used mountain pass theorem [2] and Ekeland principle [1].

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THE GLOBAL EXISTENCE AND NUMERICAL SIMULATION FOR A COUPLED REACTION-DIFFUSION SYSTEMS ON EVOLVING DOMAINS

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ABSTRACT. The aim of this paper is to demonstrate the global existence, uniqueness and uniform boundedness of solutions for a weakly coupled class of reaction-diffusion systems on isotropically growing domain. Our results generalize some known results on fixed domains and in addition to the new results on evolving domains. As well as we affirm our theoritical findings through numerical exprements.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 35R37, 35A01, 81T80.

KEYWORDS AND PHRASES. Reaction-diffusion systems, global existence, Lyapunov function.

1. DEFINE THE PROBLEM

Let $\Omega_t \subset \mathbb{R}^N (N \geq 1)$ be a simply connected, bounded, time-dependent domain with its moving boundary $\partial\Omega_t$ is smooth ($t \geq 0$), which can be mapped into a static reference domain Ω_0 by using a C^k-diffeomorfism ($k \geq 2$) $\Xi_t : \Omega_0 \to \Omega_t$. Moreover, the diffeomorfism Ξ_t are assumed belongs to the class C^2 with respect to the variable t. The evolution equations for reaction-diffusion systems can be obtained from the application of the law of mass conservation in an elemental volume using Reynolds transport theorem. The change of domain's volume Ω_t generates a flow of velocity $\vartheta(x,t)$. Therefore, the evolving of domain has the effect of introducing the following extra terms to the classical model of reaction-diffusion, an advection term $\vartheta.\nabla u$ represents the transport of the species u by the flow velocity ϑ and a dilution term $u(\nabla.\vartheta)$ due to local volume change (cf. [1]). In this work we deal with a class of reaction-diffusion systems on a growing domain which takes the following form:

$$
(1) \quad \begin{cases}
\dfrac{\partial u}{\partial t} + \nabla.(\vartheta u) - d_1 \Delta u = \Lambda - \lambda(t) f(u,v) - \mu u & \text{in } \Omega_t \times (0,T)\,, \\
\dfrac{\partial v}{\partial t} + \nabla.(\vartheta v) - d_2 \Delta v = \lambda(t) f(u,v) - \sigma h(v) & \text{in } \Omega_t \times (0,T)\,, \\
\dfrac{\partial u}{\partial \nu}(x,t) = \dfrac{\partial v}{\partial \nu}(x,t) = 0 & \text{on } \partial\Omega_t \times (0,T)\,, \\
u(y,0) = u_0(y) \geqslant 0,\ v(y,0) = v_0(y) \geqslant 0 & \text{on } \overline{\Omega}_0,
\end{cases}
$$

where $T > 0$, $x := x(t) = (x_1(t), \ldots, x_N(t))$, with ν being the unit outer normal to $\partial\Omega_t$, $d_1, d_2, \mu, \sigma > 0$, $\Lambda \geqslant 0$, and $\lambda \in C^1(\mathbb{R}_+;\mathbb{R}_+)$. All along the paper, we will use the following assumptions:

(A1) $f \in C^1(\mathbb{R}_+^2;\mathbb{R}_+)$, $h \in C^1(\mathbb{R}_+;\mathbb{R}_+)$, $f(0,\eta) = h(0) = 0$ for all $\eta \in \mathbb{R}_+$.

(A2) The flow velocity $\vartheta(x,t)$ is identical to the domain velocity, i.e., $\vartheta = \dfrac{\mathrm{d}x}{\mathrm{d}t}$.

(A3) Isotropic domain deformation, i.e., the diffeomorfism Ξ_t satisfies

$$x = \Xi_t(y) = \chi(t)y, \quad y \in \Omega_0, \quad x \in \Omega_t, \quad t \in [0,T]. \tag{2}$$

(A4) $\chi \in C^2\left(\mathbb{R}_+; \mathbb{R}_+^*\right)$, $\chi(0) = 1$, $\inf\limits_{t\geqslant 0} \chi(t) > 0$, and

$$N \inf_{t\geqslant 0} \frac{\mathrm{d}\chi(t)}{\mathrm{d}t} > -\min\{\mu,\sigma\} \inf_{t\geqslant 0} \chi(t) \tag{3}$$

(A5) There exist a nondecreasing function $\varphi \in C^1(\mathbb{R}_+;\mathbb{R}_+)$ and $g \in C^1\left(\mathbb{R}_+^2;\mathbb{R}_+\right)$, such that

$$f(\xi,\eta) \leq \varphi(\xi)g(\xi,\eta), \quad \forall(\xi,\eta) \in \mathbb{R}_+^2, \quad \lim_{\eta\to\infty} \frac{log(1+g(.,\eta))}{\eta} = 0. \tag{4}$$

(A6) $h(\eta) - \eta \geqslant 0$ for all $\eta \in \mathbb{R}_+$, and $\lim\limits_{\eta\to\infty} \dfrac{log(1+h(\eta))}{\eta} = 0$.

The main purpose of this study is to supplement the investigations of [12, 13]. We prove the global existence, uniqueness and uniform boundedness of solutions for system (1) on domains with isotropic growth, and nonlinearities of weak exponential growth.

Remark 1.1. *From the assumptions (A2)-(A4), the flow velocity ϑ has the following explicit form*

$$\vartheta(x,t) = \frac{\dot{\chi}(t)}{\chi(t)}x, \quad x \in \Omega_t, \quad t \in [0,T] \tag{5}$$

where $\dot{\chi}(t) = \dfrac{\mathrm{d}\chi(t)}{\mathrm{d}t}$. Thus, the divergence of the flow velocity ϑ takes the form, $\nabla.\vartheta = N\dfrac{\dot{\chi}(t)}{\chi(t)}$.

Remark 1.2. *From the assmption (A4) we note that, if the domain growth function is evolving increasingly (e.g. logistic growth) then the parameters μ and σ are arbitrary in $\mathbb{R}_+^* := (0,+\infty)$.*

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UNSTEADY FLOW OF BINGHAM FLUID IN A THIN LAYER WITH MIXED BOUNDARY CONDITIONS

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Abstract. In this paper we consider the dynamic system for Bingham fluid in a three-dimensional thin domain with Fourier and Tresca boundary condition. We study the existence and uniqueness results for the weak solution, then we establish its asymptotic behavior, when the depth of the thin domain tends to zero. This study yields a mechanical laws that give a new description of the behavior this system.

2010 Mathematics Subject Classification. 35B40, 47A52, 76D20.

Keywords and phrases. Mixed boundary problems, Bingham fluid, Lubrication problem, A priori estimates.

1. Define the problem

This work gives an extension to describe the flow of fluids in a dynamic system to some of the results obtained in a series of papers [1, 2, 3], in which the authors considered a stationary case only of the general equations describing the motion of some fluid flows in bounded thin domain, with slip and mixed boundary conditions. The aim of this paper is to study the asymptotic analysis of an incompressible Bingham fluid in a dynamic regime in a three dimensional thin domain mixed boundary and subject to slip phenomenon on a part of the boundary. We are interested here in the existence and uniqueness for this problem and also its behavior when the thickness of the thin domain tends to zero. The departure point is the laws of conservation, which includes here the effect of the acceleration-dependent inertia forces. A friction law of Tresca and the Fourier boundary condition are assumed on the boundary. Then we will compare our results to stationary problem in [2, 3, 4].

The main difficulty here is to estimate the solutions of the problem, due to the fractional term for the Bingham constitutive law and the assumption coming from the initial velocity. The proofs presented in this work are based on regularization methods and classical results for elliptic variational. We present in first, some notation and the weak formulation of problem. Second ,we introduce a scaling as in [4], we give some needed estimates on the velocity and pressure, also the convergence results. Finally, we present the limit problem and we give the mechanical interpretation of the results.

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VARIABLE HERZ-TYPE HARDY ESTIMATE OF MARCINKIEWICZ INTEGRALS OPERATORS

RABAH HERAIZ

Abstract. In this communication, we present two results concerning the Marcinkiewicz integral operator μ. In the first, we show that μ is bounded from $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$ for $\alpha(\cdot), p(\cdot)$ and $q(\cdot)$ satisfies some conditions. Next, we present the boundedness of μ on variable Herz-type Hardy spaces $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(\mathbb{R}^n)$, these results are from [2].

2010 Mathematics Subject Classification. 46E30, 42B35, 42B20, 42B25.

Keywords and phrases. Marcinkiewcz integral operators, Variable Herz spaces, Variable Herz-type Hardy spaces

1. Variable Herz-type Hardy Estimate of Marcinkiewicz Integrals Operators

For $0 < \beta \leq 1$, the Lipschitz space $\mathrm{Lip}_\beta(\mathbb{R}^n)$ is defined as

$$\mathrm{Lip}_\beta(\mathbb{R}^n) := \left\{ f : \|f\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} = \sup_{x,y\in\mathbb{R}^n; x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty \right\}.$$

Given $\Omega \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ be a homogeneous function of degree zero and

$$\int_{\mathrm{S}^{n-1}} \Omega\left(x'\right) \, \mathrm{d}\sigma(x') = 0$$

where $x' = x/|x|$ for any $x \neq 0$ and S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure.

The Marcinkiewicz integral μ is defined by

$$\mu(f)(x) := \left(\int_0^\infty \left| F_\Omega f(x) \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{\frac{1}{2}}$$

where

$$F_\Omega f(x) := \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) \, \mathrm{d}y.$$

It is well known that the operator μ was first defined by Stein [1] and under the conditions above, Stein proved that μ is of type (p,p) for $1 < p \leq 2$ and of weak type $(1,1)$.

We define the set of variable exponents by

$$\mathcal{P}_0(\mathbb{R}^n) := \{p \text{ measurable: } p(\cdot) : \mathbb{R}^n \to [c, \infty[\text{ for some } c > 0\}.$$

The subset of variable exponents with range $[1, \infty)$ is denoted by $\mathcal{P}(\mathbb{R}^n)$. For $p \in \mathcal{P}_0(\mathbb{R}^n)$, we use the notation

$$p^- = ess \inf_{x \in \mathbb{R}^n} p(x), \quad p^+ = ess \sup_{x \in \mathbb{R}^n} p(x).$$

Definition 1.1. Let $p \in \mathcal{P}_0(\mathbb{R}^n)$. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the class of all measurable functions f on $\mathbb{R}^n$ such that the modular

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx$$

is finite. This space is a quasi-Banach function space equipped with the norm

$$\|f\|_{p(\cdot)} := \inf\left\{\mu > 0 : \varrho_{p(\cdot)}\left(\frac{1}{\mu} f\right) \leq 1\right\}.$$

If $p(x) \equiv p$ is constant, then $L^{p(\cdot)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ is the classical Lebesgue space.

Definition 1.2. We say that a function $g : \mathbb{R}^n \to \mathbb{R}$ is locally log-Hölder continuous, if there exists a constant $c_{\log} > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}}{\ln(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}^n$. If

$$|g(x) - g(0)| \leq \frac{c_{\log}}{\ln(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$, then we say that g is log-Hölder continuous at the origin (or has a log decay at the origin). If, for some $g_\infty \in \mathbb{R}$ and $c_{\log} > 0$, there holds

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\ln(e + |x|)}$$

for all $x \in \mathbb{R}^n$, then we say that g is log-Hölder continuous at infinity (or has a log decay at infinity).

Definition 1.3. Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf\left\{\lambda_v > 0 : \varrho_{p(\cdot)}\left(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\right) \leq 1\right\}.$$

The (quasi)-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\left\{\gamma > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{1}{\gamma}(f_v)_v\right) \leq 1\right\}.$$

Since $q^+ < \infty$, then we can replace by the simpler expression $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum\limits_v \left\| |f_v|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}$.

If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function. Before giving the definition of variable Herz spaces, let us introduce the following notations

$$B_k := B(0, 2^k), \quad R_k := B_k \setminus B_{k-1} \text{ and } \chi_k = \chi_{R_k}, \quad k \in \mathbb{Z}.$$

Definition 1.4. Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The inhomogeneous Herz space $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L^{p(\cdot)}_{Loc}(\mathbb{R}^n)$ such that

$$\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|f\,\chi_{B_0}\|_{p(\cdot)} + \left\|\left(2^{k\alpha(\cdot)} f\,\chi_k\right)_{k\geq 1}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Similarly, the homogeneous Herz space $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in L^{p(\cdot)}_{Loc}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} := \left\|\left(2^{k\alpha(\cdot)} f\,\chi_k\right)_{k\in\mathbb{Z}}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

The Hardy-Littlewood maximal operator M is defined on L^1_{loc} by

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ and radius $r > 0$. It was shown that $M : L^{p(\cdot)} \to L^{p(\cdot)}$ is bounded if $p \in \mathcal{P}^{\log}$ and $p^- > 1$.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subseteq B_0$, $\int_{\mathbb{R}^n}\varphi(x)dx \neq 0$ and $\varphi_t(\cdot) = t^{-n}\varphi\left(\frac{\cdot}{t}\right)$ for any $t > 0$. Let $M_\varphi(f)$ be the grand maximal function of f defined by

$$\mathcal{M}_\varphi(f)(x) := \sup_{t>0} |\varphi_t * f(x)|.$$

Here we give the definition of the homogeneous Herz-type Hardy spaces $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}$.

Definition 1.5. Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The homogeneous Herz-type Hardy space $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\mathcal{M}_\varphi(f) \in \dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ and we define

$$\|f\|_{H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}} := \|\mathcal{M}_\varphi(f)\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}}.$$

2. Mains results

In this section, we present two results concerning the Marcinkiewicz integral operator μ. In the first, we show that μ is bounded from $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ for $\alpha(\cdot), p(\cdot)$ and $q(\cdot)$ satisfies some conditions.

Theorem 2.1 ([2])**.** *Suppose that $0 < \tau \leq 1, p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^+ < \infty, \Omega \in L^s(S^{n-1}), s > (p')^-$ and $\alpha \in L^\infty(\mathbb{R}^n)$, $q \in \mathcal{P}_0(\mathbb{R}^n)$. If α and q have a* log *decay at the origin such that*

$$-\frac{n}{p(0)} - \frac{n}{s} - \tau < \alpha(0) < n - \frac{n}{p(0)} - \frac{n}{s} - \tau \text{ and } -\frac{n}{p_\infty} - \frac{n}{s} - \tau < \alpha_\infty < n - \frac{n}{p_\infty} - \frac{n}{s} - \tau$$

then μ is bounded from $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ (or $K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$) to $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ (or $K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$)

In the next result we treat the boundedness of Marcinkiewicz integral operators with homogeneous kernel on variable Herz-type Hardy spaces.

Theorem 2.2 ([2]). *Suppose that* $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{R}^n)$ *with* $p_1^+ < 2n$ *and* $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{1}{2n}$, $\alpha \in L^\infty(\mathbb{R}^n)$, $q_1, q_2 \in \mathcal{P}_0(\mathbb{R}^n), \Omega \in L^s(S^{n-1})$ *with* $s > (p_1')^-$. *If* α, q_1 *and* q_2 *are* log*-Hölder continuous, both at the origin and at infinity such that*

$$\alpha(\cdot) \geq n(1 - \frac{1}{p_1^-}), q_1(0) \leq q_2(0) \text{ and } (q_1)_\infty \leq (q_2)_\infty .$$

Then μ *is bounded from* $H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ *to* $\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(\mathbb{R}^n)$.

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Weak Solutions for the p(x)-Laplacian Equation with Variable Exponents and Irregular Data

Communication Info

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Abstract

In this presentation, we give a result of regularity of weak solutions for a class of p(x)-Laplacian equation with p(x) growth conditions, and measure or L^m data, with m>1 being small.

The functional setting involves Lebesgue-Sobolev spaces with variable exponents.

Keywords:
(1) p(x)-Laplacian Equation,
(2) Weak Solution,
(3) Variable Exponents,
(4) Irregular Data.

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STABILITY OF A HIGH-ORDER Q-FRACTIONAL SYSTEM

LOUIZA TABHARIT AND HOUARI BOUZID

ABSTRACT. This work is devoted to the study of existence and uniqueness of the solution of the q-fractional differential system (1). For this purpose, we use a fixed point theorem of Banach. The stability in the sense of " Ulam-Hyers Rassias " of the solution with respect to the initial integro-differential conditions is proved. Besides, we discuss an example for illustration of the main work.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 26A33, 34K20, 39B72.

KEYWORDS AND PHRASES. Caputo derivative, fractional integral, q-analogue, fixed point, stability .

1. Define the problem

Recently, the q-fractional calculus has gained a lot of attention. Considered as a relationship between mathematics and physics, it derives its importance from the fact that it intervenes in distinguished fields such as quantum mechanics and stochastic analysis, chemistry and neurology [1], [2], [3]. With a wide expansion of fractional calculus, the study of the stability of fractional differential equations has also motivated researchers to produce many contributions [4], [5], [6]. Our new results are essentially based on the following nonlinear fractional system of differential equations, for $t \in [0;1]$:

$$\begin{cases} D_q^{\alpha_1} u_1(t) = C_1(t) f_1(t, u_1(t), ..., u_m(t)) + \sum\limits_{i=1}^{l} g_i^1(t, D_q^{\gamma} u_1(t), ..., D_q^{\gamma} u_m(t)) \\ D_q^{\alpha_2} u_2(t) = C_2(t) f_2(t, u_1(t), ..., u_m(t)) + \sum\limits_{i=1}^{l} g_i^2(t, D_q^{\gamma} u_1(t), ..., D_q^{\gamma} u_m(t)) \\ \vdots \\ D_q^{\alpha_m} u_m(t) = C_m(t) f_m(t, u_1(t), ..., u_m(t)) + \sum\limits_{i=1}^{l} g_i^m(t, D_q^{\gamma} u_1(t), ..., D_q^{\gamma} u_m(t)) \\ u_k(0) = \tau_k, \\ u_k^{(j)}(0) = 0, j = 1, ..., n-2 \\ D_q^{n-\alpha_k} u_k(1) = J_q^{\alpha_k - n + 1} u_k(\lambda_k), \end{cases}$$

(1)

where $D_q^{\alpha_k}$ denote the q-derivative of Caputo and $J_q^{\alpha_k-n+1}$ is the q-fractional integral, $0 < q < 1, l, m \in \mathbb{N}^*$, $\alpha_k \in (n-1, n]$, $\gamma \in (0, n-1]$, $\tau_k, \lambda_k \in \mathbb{R}^+$ and f , $g : [0,1] \times \mathbb{R}^m \to \mathbb{R}$, $C_k : I \to \mathbb{R}$ are given functions.

We define the Banach space in which we will study the uniqueness of the solution by

$$S := \{(u_1, ..., u_m) : u_k \in C([0,1], \mathbb{R}),\ D^{\alpha_k} u_k \in C([0,1], \mathbb{R})\ \ ,\ \ k = 1, 2, ..., m\};$$

endowed with the norm :

$$\|(u_1, ..., u_m)\|_S = \max_{1 \le k \le m} (\|u_k\|_\infty, \|D^{\alpha_k} u_k\|_\infty) \text{ where } \|u_k\|_\infty = \sup_{0 \le t \le 1} |u_k|,\ k = 1, 2, ..., m.$$

By the following lemma, we present the integral form of the solution of our system.

Lemma 1.1. *The solution of the fractional differential equation*

$$D_q^{\alpha_k} u_k(t) = h_k(t), \qquad n - 1 < \alpha_k < n,\ \ 0 < q < 1,\ n \in \mathbb{N} - \{1\}; \tag{2}$$

with $(h_k)_{k=1,...,m} \in C([0,1], \mathbb{R})$, *under the following conditions*

$$\begin{cases} u_k(0) = \tau_k, \\ u_k^{(j)}(0) = 0,\ j = 1, ..., n-2 \\ D_q^{n-\alpha_k} u_k(1) = J_q^{\alpha_k - n + 1} u_k(\lambda_k) \end{cases} ; k = 1, ..., m,$$

is given by the formula

$$\begin{aligned} u_k(t) &= \frac{1}{\Gamma_q(\alpha_k)} \int_0^t (t - qs)^{(\alpha_k - 1)} h_k(s) d_q s + \tau_k \\ &+ t^{n-1} \left(\frac{\Gamma_q(\alpha_k + 1)}{\left(\lambda_k^{\alpha_k} - [\alpha_k]_q\right) \Gamma_q(n)} \right) \int_0^1 \frac{(1 - qs)^{(2\alpha_k - n - 1)}}{\Gamma_q(2\alpha_k - n)} h_k(s) d_q s \\ &- t^{n-1} \left(\frac{\Gamma_q(\alpha_k + 1)}{\left(\lambda_k^{\alpha_k} - [\alpha_k]_q\right) \Gamma_q(n)} \right) \int_0^{\lambda_k} \frac{(\lambda_k - qs)^{(2\alpha_k - n)}}{\Gamma_q(2\alpha_k - n + 1)} h_k(s) d_q s \\ &- t^{n-1} \left(\frac{\Gamma_q(\alpha_k + 1)}{\left(\lambda_k^{\alpha_k} - [\alpha_k]_q\right) \Gamma_q(n)} \right) \frac{\tau_k \lambda_k^{\alpha_k - n + 1}}{\Gamma_q(\alpha_k - n + 2)}, \end{aligned}$$

where $\lambda_k^{\alpha_k} \neq [\alpha_k]_q$.

Moreover, under some hypothisis we show uniqueness of the solution of the nonlinear system (1) by:

Theorem 1.2. *Assume that* $(H1)$ *and* $(H3)$ *are satisfied. If*

$$\max_{0 \le k \le m} \left(M_k \omega_k +_{i=1}^{l} \varphi_i^k \right) \left(\Delta_1^k, \nabla_1^k \right) < 1$$

is valid, then $(P1)$ *has a unique solution on* $[0, 1]$.

Our second main result consists to prove that the solution is stable in the sens of Ulam-Hyers-Rassias.

Theorem 1.3. *Assume that* $(H1)-(H3)$ *and* $\left(M_k\omega_k +_{i=1}^{l} \varphi_i^k\right) < 1$ *holds. If there exists* $\Phi \in C\left([0,1], \mathbb{R}^+\right)$ *such that*

$$\left|D_q^{\alpha_k} u_k\left(t\right) - C_k(t) f_k(t, u_1(t), ..., u_m(t)) -_{i=1}^{l} g_i^k(t, D_q^{\gamma} u_1(t), ..., D_q^{\gamma} u_m(t))\right| \leq \epsilon_k \Phi\left(t\right),$$

is valid for $k = 0, ..., m; t \in [0,1]$*, then the fractional system* $(P1)$ *is Ulam-Hyers-Rassias stable with respect to* Φ*.*

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Modeling and numerical analysis

A COMPACT FOURTH ORDER FINITE DIFFERENCE SCHEME FOR THE DIFFUSION EQUATION WITH NONLINEAR NONLOCAL BOUNDARY CONDITIONS

S.DEHILIS, A.BOUZIANI, AND S.BENSAID

Abstract. In this article, fourth-order compact finite difference scheme is developed to solve the diffusion equation with nonlinear nonlocal boundary conditions. The proposed scheme is derived by combining a fourth-order compact finite difference formula in space and a backward differentiation for the time derivative term. Nonlinear terms are linearized by Taylor expansion. Numerical examples are provided to verify the accuracy and efficiency of our proposed method.

2010 Mathematics Subject Classification. 35K58, 65L12.

Keywords and phrases. Nonlinear nonlocal boundary conditions, Fourth-order compact difference scheme.

1. Define the problem

Consider the diffusion equation in one-dimensional time-dependent

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \qquad 0 < x < 1, \qquad 0 < t \leq T, \tag{1}$$

with the initial condition

$$u(x,0) = \phi(x), \qquad 0 < x < 1, \tag{2}$$

and the nonlinear nonlocal boundary conditions

$$u(0,t) = \int_0^1 p(x,t)\varphi(u(x,t))dx + E(t), \qquad 0 < t \leq T, \tag{3}$$

$$u(1,t) = \int_0^1 q(x,t)\psi(u(x,t))dx + G(t), \qquad 0 < t \leq T, \tag{4}$$

where $f, p, q, \phi, \varphi, \psi, G$ and E are known functions.

Mathematical formulation of this problem arises naturally in various engineering models, such as thermoelasticity ([4] thermodynamics [5], heat conduction [2, 3, 1, 6]. Many numerical methods in the past few years have been developed for solving a parabolic initial-boundary value problems which involve nonlocal boundary conditions of the type :

$$u(0,t) = \int_0^1 p(x,t)\,u(x,t)\,dx + E(t), \qquad 0 < t \leq T,$$

$$u(1,t) = \int_0^1 q(x,t)\,u(x,t)\,dx + G(t), \qquad 0 < t \leq T.$$

Much less effort is given to the problem with nonlinear nonlocal type boundary conditions (3) and (4). Authors of [1] considered the implicit difference scheme for the solution of the heat equation with nonlinear nonlocal boundary condition of the type :

$$u(0,t) = \int_0^1 p(x,t)\, u^{\gamma}(x,t)\, dx + E(t), \qquad 0 < t \leq T, \tag{5}$$

$$u(1,t) = \int_0^1 q(x,t)\, u^{\gamma}(x,t)\, dx + G(t), \qquad 0 < t \leq T, \tag{6}$$

Recently, the authors of [2] proposed a second order accurate difference scheme for the diffusion equation with nonlinear nonlocal boundary conditions (5) and (6), authors used the Forward time centred space (FTCS), DufortFrankel scheme (DFS), Backward time centred space (BTCS), Crank-Nicholson method (CNM).

Therefore this work is aimed at producing a fourth order accurate difference scheme for the diffusion equation with nonlinear nonlocal boundary conditions (3)and (4).

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A DISCRETISED APPROACH FOR A PDE-CONSTRAINED BI-OBJECTIVE OPTIMAL CONTROL PROBLEM

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Abstract. In this paper, we are interesting in solving numerically the optimal control problem governed by an advection-diffusion equation which model a practical environmental problem. The infinite dimensional problem is discretized by application of discontinuous Galerkin method. Then, we discretized the objective and the PDE equation.

2010 Mathematics Subject Classification. 65N30, 49J20, 65K10, 49J52.

Keywords and phrases. Optimal control problem, discontinuous Galerkin method, advection-diffusion equation, euler backward.

1. Define the problem

We try to solve the following state constrained optimal control problem of linear steady advection-diffusion equation:

$$\min_{y}\left\{\frac{1}{2}\int_0^T\int_\Omega [y(t,x)-y_d(t,x)]^2dxdt,\ \frac{1}{2}\int_0^T ||u(t)||^2dt\right\} \tag{1}$$

(2)
$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x)-k\Delta_x y+\beta(t,x).\nabla y(t,x)=\displaystyle\sum_{i=1}^m u_i(t)\chi_i(x) \quad \text{for } (t,x)\in[0,T]\times\Omega \\ \dfrac{\partial y}{\partial \eta}(t,x)+\alpha_i y(t,x)=\alpha_i y_a(t) \text{ for } (t,x)\in\Sigma_i=(0,T)\times\Gamma_i \\ y(0,x)=y_0(x) \text{ for } x\in\Omega. \end{cases}$$

and

$$u_a(t)\le u(t)\le u_b(t),\ \text{for almost everything } t\in[0,T]. \tag{3}$$

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A NUMERICAL SOLUTION FOR A COUPLING SYSTEM OF CONFORMABLE TIME-DERIVATIVE TWO DIMENSIONAL BURGERS'EQUATIONS

ILHEM MOUS1 AND ABDELHAMID LAOUAR2

Abstract. In this paper, we deal with a numerical solution for a coupling system of fractional conformable time-derivative two-dimensional (2D) Burgers equations. The presence of both the fractional time derivative and the nonlinear terms in this system of equations makes solving it more difficult. Firstly, we use the Cole-Hopf transformation in order to reduce the coupling system of equations to a conformable time-derivative 2D heat equation for which the numerical solution is calculated by the explicit and implicit schemes. Secondly, we calculate the numerical solution of the proposed system by using both the obtained solution of the conformable time-derivative heat equation and the inverse Cole-Hopf transformation. This approach may show its efficiency to deal with this class of fractional nonlinear problems. Some numerical experiments are displayed to consolidate our approach.

2010 Mathematics Subject Classification. 34A08, 26A33, 34K28.

Keywords and phrases. Burgers equation, Cole-Hopf transformation, Conformable time-derivative.

1. Define the problem

In this work, we are interested in studying a following coupling system of the fractional conformable derivative 2D Burgers'equations which incorporate the interaction between the nonlinear convection processes and the diffusive viscous processes

$$\text{(1)} \quad \begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = r\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right), \\[2ex] \dfrac{\partial^\alpha v}{\partial t^\alpha} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = r\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right), \end{cases}$$

where $\alpha \in]0;1[$, $r > 0$ the diffusion coefficient, $(x, y) \in \Omega$ (a rectangular domain), $t > 0$ and $\partial^\alpha u/\partial t^\alpha$, $\partial^\alpha v/\partial t^\alpha$ mean conformable derivatives respectively of the functions $u(x, y, t)$ and $v(x, y, t)$.

Subject to the initial conditions

$$\text{(2)} \quad \begin{cases} u(x, y, 0) = u_0(x, y), \text{ for any } (x, y) \in \Omega, \\[2ex] v(x, y, 0) = v_0(x, y), \text{ for any } (x, y) \in \Omega, \end{cases}$$

and the boundary conditions

$$
\left\{
\begin{array}{l}
u(x,y,t) = f(x,y,t), \text{ for any } (x,y) \in \partial\Omega, \ t > 0, \\
\\
v(x,y,t) = g(x,y,t), \text{ for any } (x,y,t) \in \partial\Omega, \ t > 0,
\end{array}
\right. \tag{3}
$$

where f, g are two given functions.

We need later to use the following potential symmetry condition

$$
\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{4}
$$

Many works concerned the one/two viscous Burgers'equation (with integer-order derivative) using the Cole-Hopf transformation [2, 5]. It is known that the Burgers'equation has been used as a mathematical model in various areas such as number theory, gas dynamics, heat conduction, elasticity theory, etc. It has a lot of similarity to the famous Navier-Stokes equations [1, 3] and has often been used as a simple model equation for comparing the accuracy of different computational algorithms. However the inviscid Burgers equation lacks one most important property attributed to turbulence since the solutions do no exhibit chaotic features like sensitivity with respect to initial conditions.

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A NEW DECOMPOSITION APPROACH BY EIGENVALUES FOR APPLICATION OF DIFFERENCE OF CONVEX FUNCTIONS ALGORITHM IN SOLVING QUADRATIC PROBLEMS

SAADI ACHOUR

Abstract. Difference of Convex functions Algorithms (DCA) are used to solve nonconvex optimization problems, specifically quadratic programming ones, generally by finding global approximate solutions expediently. DCA efficiency depends on two basic parameters that directly affect the speed of its convergence towards the optimal solution. The first parameter is the selected decomposition and the second is the assigned initial point. In this work, I propose a new decomposition for a DCA in quadratic form. The proposed decomposition uses the Eigenvalues of the matrix of the quadratic part of the problem, herein named as Quadratic Decomposition for the Difference of Convex functions by Eigenvalues ($QDDCE$). In order to test the performance of $QDDCE$, I propose an experimental study using a set of nonconvex quadratic problems based on an implementation framework with MATLAB to allow assessment of key performance indicators (including the computing time and possible dimensions of the problems). The Results demonstrated the possibility of applying the $QDDCE$ for problems with $n \leq 40$ dimensions, while difficulties were experienced with problems of $n > 40$ dimensions. The reason for this complexity is the difficulty of computing the Eigenvalues in large dimensions. Note that these results were obtained using a computer with medium specifications, and therefore the number of dimesions should increase with a higher performance machine. To conclude, this work proposes a decomposition strategy using Eigenvalues, which should facilitate application of DCA to nonconvex quadratic problems.

2010 Mathematics Subject Classification. 90C26, 90C27, 90C20.

Keywords and phrases. Nonconvex quadratic programming, Numerical experiments, Approximated global minium, DCA, Matlab, .

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A PRIMAL-DUAL INTERIOR POINT METHOD FOR HLCP BASED ON A CLASS OF PARAMETRIC KERNEL FUNCTIONS

NADIA HAZZAM AND ZAKIA KEBBICHE

Abstract. In an attempt to improve theoretical complexity of large-update methods, in this paper, we propose a primal-dual interior-point method for $P_*(\kappa)$-horizontal linear complementarity problem. The method is based on a class of parametric kernel functions. We show that the corresponding algorithm has the best known iteration bounds for large-update methods for $P_*(\kappa)$-horizontal linear complementarity problem that is $O\left((1+2\kappa)\sqrt{n}\log n \log \frac{n}{\epsilon}\right)$. We illustrate the performance of the proposed kernel function by some comparative numerical results that are derived by applying our algorithm on five kernel functions.

2010 Mathematics Subject Classification. 90C33; 90C51.

Keywords and phrases. Horizontal linear complementarity problem, $P_*(\kappa)$-matrix, interior-point method, kernel function, complexity bound.

1. Define the problem

The aim of our paper is to propose a primal-dual interior-point method based on a class of trigonometric kernel functions for solving the horizontal linear complementarity problem (HLCP) in the standard form

$$-Mx + Ny = q, \quad xy = 0, \quad (x, y) \geq 0, \tag{1}$$

where $M, N \in \mathbb{R}^{n\times n}, q \in \mathbb{R}^n$ and xy denotes the componentwise product of vectors x and y.

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A Taylor Collocation Method to Solve Integro-differential Equations of the second kind

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Abstract

In this paper, we present Taylor Collocation method to find approximate solution for integro-differential equation.In which we transform the differential part using the backward difference and the integral part to a matrix form. In the end, we provide an error analysis and we conclude by giving the algorithm.

Keywords: Fredholm integro-differential equation, Taylor Collocation, Backward Difference.

A FINITE VOLUME METHOD FOR THE DARCY PROBLEM

AKRAM BOUKABACHE

Abstract. The goal of this presentation is to introduce a simple finite volume method to solve the Darcy problem. This method is so simple such that both velocity and pressure are approximated by piecewise constant functions, and the stability of the scheme is obtained by adding to the mass balance stabilization terms

2010 Mathematics Subject Classification. 76D07, 65M08

Keywords and phrases. finite volume methods, Darcy problem,

1. Define the problem

The Darcy equations can be written as:

$$\alpha\mathbf{u}+\nabla p = \mathbf{f} \quad \text{in } \Omega \tag{1a}$$
$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \tag{1b}$$
$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega \tag{1c}$$

where $\mathbf{u}$ can be interpreted as the velocity field of an incompressible fluid motion, p is then the associated pressure and α is a positive constant.

The weak formulation of the Darcy equations seeks $(\mathbf{u},p) \in \mathbf{H}_0(\text{div};\Omega) \times L^2_0(\Omega)$ such that

$$\alpha(\mathbf{u},\mathbf{v})_{0,\Omega} - (p,\nabla\cdot\mathbf{v})_{0,\Omega} = (\mathbf{f},\mathbf{v})_{0,\Omega} \qquad \forall\mathbf{v}\in\mathbf{H}_0(\text{div};\Omega) \tag{2a}$$
$$(q,\nabla\cdot\mathbf{u})_{0,\Omega} = 0 \qquad \forall q\in L^2_0(\Omega) \tag{2b}$$

we get the following

Theorem 1.1. *The weak Darcy problem has a unique solution* $(\mathbf{u},p) \in \mathbf{H}_0(\text{div};\Omega)\times L^2_0(\Omega)$.

For the construction of the discrete problem the Galerkin method is followed. let $\mathbf{X}_h \subset \mathbf{H}_0(\text{div};\Omega)$ and $M_h \subset L^2_0(\Omega)$ be two finite-dimensional spaces with h the discretization parameter.

Following the Petrov-Galerkin methodology and a stabilization procedure, a discrete formulation reads

$$\begin{aligned}
\alpha(\mathbf{u}_h,\mathbf{v}_h)_{0,\Omega} - (p_h,\nabla_h\cdot\mathbf{v}_h)_{0,\Omega} &= (\mathbf{f},\mathbf{v}_h)_{0,\Omega} & \forall\mathbf{v}_h\in\mathbf{X}_h\\
(q_h,\nabla_h\cdot\mathbf{u}_h)_{0,\Omega} + J(p_h,q_h) &= 0 & \forall q\in X_h
\end{aligned}$$

where

$$J(p_h,q_h) = \delta\sum_K \int_{\partial K\setminus\partial\Omega} h_{\partial K}\,[p_h]\,[q_h]ds$$

is a stabilization term, with $\delta > 0$.

Lemma 1.2 (Scheme stability)**.** *There exists two positive real numbers c_1 and c_2 independent of h such that, for all $p_h \in M_h$ one can find $\mathbf{v}_h \in \mathbf{X}_h$ satisfying:*

$$\|\mathbf{v}_h\|_{\mathcal{D}} = 1$$
$$b(\mathbf{v}_h, p_h) \geq c_1 \|p_h\|_{0,\Omega} - c_2 h |v_h|_{\mathcal{D}}$$

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A FRICTIONAL CONTACT PROBLEM BETWEEN TWO PIEZOELECTRIC BODIES WITH NORMAL COMPLIANCE CONDITION AND ADHESION

TEDJANI HADJ AMMAR

Abstract. We consider a mathematical model which describes the quasistatic frictional contact problem between two piezoelectrics bodies with normal compliance condition and adhesion. The evolution of the bonding field is described by a first order differential equation. We derive variational formulation for the model and prove an existence and uniqueness result of the weak solution. The existence of a unique weak solution of the model is established under a smallness assumption of the friction coefficient. The proof is based on arguments of evolutionary variational inequalities and Banach's fixed point theorem.

2010 Mathematics Subject Classification. 35Q74, 47H10, 49J40, 74D10.

Keywords and phrases. Piezoelectric material, adhesion, existence and uniqueness, fixed point.

An elastic material with piezoelectric effect is called an electro-elastic material and the discipline dealing with the study of electro-elastic materials is the theory of electro-elasticity. Their bases were underlined by Voigt [11] who provided he first mathematical model of a linear elastic material which takes into account the interaction between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in [5, 6, 10] and, more recently, in [1]. The importance of this paper is to make the coupling of the piezoelectric problem and a frictional contact problem with adhesion between two electro–elastics bodies. The novelty in all these papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by β, it describes the point wise fractional density of adhesion of active bonds on the contact surface, and some times referred to as the intensity of adhesion. Following [3], the bonding field satisfies the restriction $0 \leq \beta \leq 1$, when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active.

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A LIMITED-MEMORY QUASI-NEWTON ALGORITHM FOR GLOBAL OPTIMIZATION VIA STOCHASTIC PERTURBATION

RAOUF ZIADI AND ABDELATIF BENCHERIF-MADANI

Abstract. In this paper, we give a new representation to the limited memory BFGS methods, and show how to use them efficiently for solving smooth global optimization problems, by considering a random perturbation following a truncated Gauss's law. Our approach is suitable for solving large-scale bound-constrained global optimization problems. Theoretical results ensure that the proposed method converges to a global minimizer almost surely. Numerical experiments are achieved on some typical test problems and comparisons with well-known methods are carried out to show the performance of our algorithm.

2010 Mathematics Subject Classification. 90C26, 90C90.

Keywords and phrases. Global optimization, Limited memory BFGS method, Stochastic perturbation, Truncated Gauss's law.

In this paper we consider the following bound-constrained global optimization problem of the form:

$$(P) \qquad \min_{x\in D} f(x)$$

where the objective function $f(x) : \mathbb{R}^n \to \mathbb{R}$ is not necessarily convex but differentiable whose gradient at point x is $\nabla f(x)$, $D = \{x \in \mathbb{R}^n | L \leq x \leq U\}$, where L and U are lower and upper bounds.

The problem (P) is of interest in many real-world applications (such economics, electronics, telecommunication and so on) involving objective functions which are differentiable but non-convex [1, 2]. Many methods for solving differentiable global optimization problems have been proposed [5], and these methods are classified into deterministic and stochastic methods. As is well known, deterministic algorithms provide a theoretical guarantee of locating the ε-global optimum. When dealing with an oscillating function in a large search space or in relatively high dimensions, deterministic exploration methods (such as DIRECT methods, the approach based on the introduction of an auxiliary function or covering methods [3, 4, 6], etc.) are not effective and can have unreasonable calculation times. Indeed, with these approaches, it is hard to obtain useful information while exploring all the regions of the feasible domain.

Stochastic algorithms such as Simulated Annealing algorithm (SA), Classification and Regression Trees (CART), Random Walk, Tabu Search (TS), Variable Neighbourhood Search (VNS) etc. [7], involve random sampling or a combination of random sampling and local search; they are theoretically well studied. They ensure the convergence to the global minimum only in probability. Unfortunately, most of them are not well suited to efficiently solve high-dimensional problems, particularly those containing more than 10 variables.

In this paper, we suggest a method for solving large-scale problems. This method is a modification of the limited memory BFGS method for bound-constrained problems and we show how to use it efficiently to deal with global optimization problems by the adjunction of a stochastic perturbation following a truncated Gauss's law. This approach leads to a stochastic descent method where the deterministic sequence generated by the limited memory BFGS method is replaced by a sequence of random variables.

Mathematical results concerning the convergence to the global minimum are established. Numerical experiments carried out on a large number of test functions show a quite promising performance of the new algorithm in comparison with some well known stochastic methods.

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A LOGARITHMIC BARRIER METHOD VIA APPROXIMATE FUNCTIONS FOR CONVEX QUADRATIC PROGRAMMING

SORAYA CHAGHOUB AND DJAMEL BENTERKI

Abstract. In this work, we consider a convex quadratic program with inequality constraints. We use a logarithmic barrier method based on some new approximate functions, these functions allow the computation of the displacement step easily and in a short time, unlike the line search method which is expensive in terms of computational volume and necessitates much time. We have developed an implementation with MATLAB and conducted numerical tests on some examples of large size. The obtained numerical results show the accuracy and the efficiency of our approach.

2021 Mathematics Subject Classification. Optimization, Numerical analysis, Operation research.

Keywords and phrases. Quadratic programming, Line search, Approximate function.

1. Position of the problem

Let us consider the following convex quadratic problem:

$$(PQ) \qquad \begin{cases} \min q(x) = \frac{1}{2}x^tQx + c^tx \\ x \in D, \end{cases}$$

where Q is a $\mathbb{R}^{n\times n}$ symmetric semidefinite matrix, $c \in \mathbb{R}^n$ and $D = \{x \in \mathbb{R}^n : Ax \geq b\}$, such that $b \in \mathbb{R}^m$ and A is a $\mathbb{R}^{m\times n}$ matrix.

We define the unconstrained perturbed problem associated to (PQ) as follows:

$$(PQr) \qquad \begin{cases} \min q_r(x) \\ x \in \mathbb{R}^n, \end{cases}$$

where $q_r : \mathbb{R}^n \to (-\infty, +\infty]$ is a barrier function defined by:

$$q_r(x) = \begin{cases} q(x) - r\sum_{i=1}^{m} \ln < e_i, Ax - b > & \text{if } \ Ax - b > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

With $(e_1, e_2, ..., e_m)$ is the canonical base in $\mathbb{R}^m$ and r is a strictly positive barrier parameter.

We use a logarithmic barrier approach to solve a series of problems (PQr), the solution of these later will converge to that of (PQ) when r tends to zero. We use the proposed approximate functions to compute the displacement step. We also promote our study by numerical tests to prove the efficiency of the technique of approximate functions and compare it with line search method.

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A MULTI-REGION DISCRETE TIME MATHEMATICAL MODELING OF THE DYNAMICS OF COVID-19 VIRUS PROPAGATION USING OPTIMAL CONTROL

BOUCHAIB KHAJJI 1, OMAR BALATIF 2, AND MOSTAFA RACHIK 1

ABSTRACT. We study in this work a discrete mathematical model that describes the dynamics of transmission of the Corona virus between humans on the one hand and animals on the other hand in a region or in different regions. Also, we propose an optimal strategy to implement the optimal campaigns through the use of awareness campaigns in region j that aims at protecting individuals from being infected by the virus, security campaigns and health measures to prevent the movement of individuals from one region to another, encouraging the individuals to join quarantine centers and the disposal of infected animals. The aim is to maximize the number of individuals subjected to quarantine and trying to reduce the number of the infected individuals and the infected animals. Pontryagin's maximum principle in discrete time is used to characterize the optimal controls and the optimality system is solved by an iterative method. The numerical simulation is carried out using Matlab. The Incremental Cost-Effectiveness Ratio was calculated to investigate the cost-effectiveness of all possible combinations of the four control measures. Using cost-effectiveness analysis, we show that control of protecting susceptible individuals, preventing their contact with the infected individuals and encouraging the exposed individuals to join quarantine centers provides the most cost-effective strategy to control the disease.
KEYWORDS AND PHRASES. Discrete mathematical model, Multi-regions, Optimal control, Covid-19, Cost-effective intervention.

1. DEFINE THE PROBLEM

Strategies used to reduce the spread covid-19 disease

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A NEW KERNEL FUNCTION BASED INTERIOR POINT ALGORITHM FOR LINEAR OPTIMIZATION

SAFA GUERDOUH, WIDED CHIKOUCHE, AND IMENE TOUIL

Abstract. The studies on the kernel function-based primal-dual interior-point algorithms indicate that a kernel function not only represents a measure of the distance between the iteration and the central path, but also plays a critical role in improving the computational complexity of an interior-point algorithm. In this work, we present a new kernel function and give the corresponding primal-dual interior point algorithm for linear optimization. We present a simplified analysis to obtain the complexity of generic interior point method based on the proximity function introduced by this kernel function.

2010 Mathematics Subject Classification. 90C05, 90C51, 90C31.

Keywords and phrases. Linear optimization, Primal-dual interior point methods, kernel functions, Complexity analysis, Large- and small-update methods.

1. Position of the problem

In this paper, we deal with the LO problem in the standard form:

$$(P)\min\{c^T x \ : \ Ax = b, \ x \geq 0\},$$

and its dual problem

$$(D)\max\{b^T y \ : \ A^T y + s = c, \ s \geq 0\},$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are given.

Without loss of generality, we assume that (P) and (D) satisfy the interior point condition (IPC), i.e., there exists (x^0, s^0, y^0) such that

$$Ax^0 = b, \ x^0 > 0, \ A^T y^0 + s^0 = c, \ s^0 > 0$$

For solving linear optimization problems, a basic scheme of the primal-dual interior-point methods (IPMs) is to follow the central path to reach an optimal solution. The central path can be obtained by solving a parametric optimization problem in terms of a barrier function with proper barrier parameters. It is well known that the use of certain kernel functions lead to significant reduction of the complexity gap between large- and small-update methods comparing to the logarithmic kernel function [4]. This was one of the main motivations of considering other kernel functions as an alternative to classical logarithmic kernel function.

The purpose of this paper is to introduce a new kernel function with logarithmic barrier term and propose a primal–dual interior point method for LO which gives better complexity bound for large-update methods compared to the complexity obtained based on the classical logarithmic kernel function [4]. The obtained iteration bound for large-update methods, namely,

$\mathcal{O}\left(n^{\frac{3}{4}}\log\frac{n}{\epsilon}\right)$, improves the classical iteration complexity with a factor $n^{\frac{1}{4}}$. For small-update methods, we derive the iteration bound $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$, which matches the currently best known iteration bound for small-update methods.

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A PRIORI AND A POSTERIORI ERROR ANALYSIS FOR A HYBRID FORMULATION OF A PRESTRESSED SHELL MODEL

REZZAG BARA RAYHANA & MERABET ISMAIL

ABSTRACT. This work deals with the finite element approximation of a prestressed shell model using a new formulation where the unknowns are described in Cartesian and local covariant basis respectively. A penalized version is then considered. We present a robust a priori error estimation. Moreover, a reliable and efficient a posteriori error estimator is also presented.

1. THE CONSTRAINED CONTINUOUS PROBLEM.

The model takes the following variational form :

$$\begin{cases} \text{Find } U=(u,r)\in\mathbb{V} \text{ such that} \\ \mathbf{a}(U,V)+\boldsymbol{a}_p(U,V)=\mathcal{L}(v), \quad \forall V=(v,s)\in\mathbb{V} \end{cases} \tag{1.1}$$

where

$$\mathbf{a}(U,V)=ta_m(u,v)+ta_t((u,r),(v,s))+\frac{t^3}{12}a_f(r,s), \quad \boldsymbol{a}_p(r,s)=\frac{t^3}{12}a_p(r,s) \ \textbf{and}\ \mathcal{L}(v)=\int_\omega f\cdot v.$$

$$\mathbb{V}=\{(v,s_i)\in H^1\left(\omega,\mathbb{R}^3\right)\times(L^2(\omega))^3 : s_\alpha\in H^1(\omega), s_3=\tilde{\gamma}_{12}(v)=\frac{1}{2}(\partial_1 v\cdot\partial_2\varphi-\partial_2 v\cdot\partial_1\varphi), \text{ a.e in }\omega \quad v|_{\Gamma_0}=s_\alpha|_{\Gamma_0}=0\} \tag{1.2}$$

The bilinear forms a_m, a_t , a_f, and a_p correspond to the membrane, transverse shear, flexural and prestress effects, respectively. The thickness t of the shell is assumed to be constant and positive [1].

2. PENALIZED VERSIONS FOR PROBLEM (1.1).

In this section, we present a penalized versions for the prestressed model (1.1). Let us consider the relaxed function space:

$$\mathbb{X}(\omega)=\{(v,s)\in H^1\left(\omega,\mathbb{R}^3\right)\times H^1(\omega)\times H^1(\omega)\times L^2(\omega), \quad v|_{\Gamma_0}=s_\alpha|_{\Gamma_0}=0\} \tag{2.1}$$

equipped with the natural norm. Let $0<\epsilon\leq 1$. We consider the following variational problem:

$$\begin{cases} \text{Find } U_\epsilon=(u_\epsilon,r_\epsilon)\in\mathbb{X} \text{ such that} \\ \mathbf{a}(U_\epsilon,V)+\boldsymbol{a}_p(r_\epsilon,s)+\epsilon^{-1}b(U_\epsilon,V)=\mathcal{L}(V), \forall V=(v,s)\in\mathbb{X}. \end{cases} \tag{2.2}$$

where, $b(U,V)=\displaystyle\int_\omega (r_3-\tilde{\gamma}_{12}(u))(s_3-\tilde{\gamma}_{12}(v))\,dx$

2000 *Mathematics Subject Classification.* 74K25, 74S05.

Key words and phrases. shell theory, finite element, hybrid formulation, a priori analysis, posteriori analysis.

3. Finite element approximation and a posteriori error analysis:

Let $(\mathcal{T}_h)_{h>0}$ be a regular affine family of triangulations which covers the domain ω. Let $\mathcal{E}_h$ be the set of edges belonging to T which are not contained in Γ_0 and $\mathcal{E}_T^1$ be the set of elements of $\mathcal{E}_h$ which are contained in $\bar{\Gamma}_1$ and let ω_T the set of elements of $(\mathcal{T}_h)$ sharing an edge with T. We introduce the finite dimensional spaces

$$\mathbb{X}_h = \{V_h = (v_h, s_h) \in (C^0(\bar{\omega})^3)^2 / V_{h|T} \in (\mathbb{P}_k(T, \mathbb{R}^3) \times \mathbb{P}_k(T)^3),\ \forall T \in \mathcal{T}_h, k \geq 1\}. \tag{3.1}$$

we consider the following discrete problem:

$$\begin{cases} \text{Find } U_h = (u_h, r_h) \in \mathbb{X}_h \text{ such that },\\ \boldsymbol{a}(U_h, V_h) + \boldsymbol{a}_p(U_h, V_h) + \epsilon^{-1} b(U_h, V_h) = \mathcal{L}(V_h), \forall V_h = (v_h, s_h) \in \mathbb{X}_h \end{cases} \tag{3.2}$$

3.1. A priori analysis.

Proposition 3.1. *There exists a unique solution $U_h \in \mathbb{X}_h$ of the problem (3.2). Moreover, this solution satisfies*

$$\|U_h\|_{\mathbb{X}} \leq C_\epsilon \|\mathcal{L}\|.$$

Assume that the solution U_ϵ of the problem (2.2) belongs to $[H^2(\omega; \mathbb{R}^3)] \times [H^2(\omega)]^2 \times [H^1(\omega)]$ then the following a priori error estimate holds:

$$\|U_\epsilon - U_h\|_{\mathbb{X}} \leq C_\epsilon h \left(\|u_\epsilon\|_{H^2(\omega;\mathbb{R}^3)} + \sum_{\alpha=1,2} \|r_\alpha\|_{H^2(\omega)} + \|r_3\|_{H^1(\omega)}. \right) \tag{3.3}$$

In order to obtain uniform estimate, we use a mixed formulation (as in [3], sec.4). Let us first introduce the following new unknown

$$\psi_\epsilon := \frac{q(U_\epsilon)}{\epsilon}$$

and the functional space $\mathbb{M} = L^2(\omega)$. Then we rewrite the continuous penalized problem (2.2) as :

$$\begin{cases} \text{Find } (U_\epsilon, \psi_\epsilon) \in \mathbb{X} \times \mathbb{M}(\omega) \quad \text{such that}\\ \tilde{\boldsymbol{a}}(U_\epsilon, V) + (\psi_\epsilon, q(V)) = \mathcal{L}(V), \quad \forall V \in \mathbb{X}\\ (q(U_\epsilon), \phi) - \epsilon(\psi_\epsilon, \phi) = 0, \qquad \forall \phi \in \mathbb{M} \end{cases} \tag{3.4}$$

where, $\tilde{\boldsymbol{a}}(\cdot,\cdot) = \boldsymbol{a}(\cdot,\cdot) + \boldsymbol{a}_p(\cdot,\cdot)$ and we consider the following discrete problem:

$$\begin{cases} \text{Find } (U_h, \psi_h) \in \mathbb{X}_h \times \mathbb{M}_h \quad \text{such that}\\ \tilde{\boldsymbol{a}}(U_h, V_h) + (\psi_h, q(V_h)) = \mathcal{L}(V_h), \quad \forall V_h \in \mathbb{X}_h\\ (q(U_h), \phi_h) - \epsilon(\psi_h, \phi_h) = 0, \qquad \forall \phi_h \in \mathbb{M}_h \end{cases} \tag{3.5}$$

where,

$$\mathbb{M}_h = \{\phi_h \in C^0(\bar{\omega}) / \phi_{h|T} \in \mathbb{P}_k(T),\ \forall T \in \mathcal{T}_h\}. \tag{3.6}$$

Corollary 3.2. *Assume that U_ϵ belongs to $H^2(\omega, \mathbb{R}^3) \times H^2(\omega)^2 \times H^1(\omega)$. Then it holds that,*

$$\|U_\epsilon - U_h\|_{\mathbb{X}} + \|\psi_\epsilon - \psi_h\|_{\mathbb{M}} \leq Ch \|U_\epsilon\|_{H^2(\omega,\mathbb{R}^3)\times H^2(\omega)^2 \times H^1(\omega)}. \tag{3.7}$$

3.2. **A posteriori analysis.**

Theorem 3.3. *Let $f \in L^2(\omega;\mathbb{R}^3)$ the following a posteriori error estimate holds between the solution U_ϵ of problem (2.2) and the solution U_h of problem (3.2).*

$$\|U_\epsilon - U_h\|_{\mathbb{X}} \leq C\left(\left(\sum_{T\in\mathcal{T}_h}(\eta_T^2+\varepsilon_T^d)^2\right)^{\frac{1}{2}} + \varepsilon_h^c\right) \tag{3.8}$$

$$\eta_T = \sum_{i=1}^{3}\eta_T^{(i)}$$

Theorem 3.4. *Let $f \in L^2(\omega;\mathbb{R}^3)$ Then, the following bounds hold.*

$$\eta_T^{(i)} \leq C\left(\|U^\epsilon - U_h\|_{\mathbb{X}} + \left(\sum_{T\in\omega_T}\left(\varepsilon_T^d\right)^2\right)^{\dfrac{1}{2}} + \varepsilon_h^c\right) \tag{3.9}$$

where, $\eta_T^{(i)}, i = 1,2,3$ are the local (interior and jump) residuals and $\varepsilon_h^c, \varepsilon_T^d$ represent the oscillations of the coefficients and the data[2].

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ABOUT SPECTRAL APPROXIMATION OF THE GENERALIZED QUADRATIC SPECTRUM

SOMIA KAMOUCHE, HAMZA GUEBBAI, AND MOURAD GHIAT

Abstract. The aim of our work is to avoid the spectral pollution appears in the approximation of quadratic spectral problems. We build a new method which is called the generalized quadratic spectrum approximation. Therefore, we prove the convergence of our method under the collectively compact convergence.

2010 Mathematics Subject Classification. 34L16, 47A10, 47A75, 45C05, 65N15, 93B60.

Keywords and phrases. generalized quadratic spectrum, spectral approximation, spectral pollution.

1. Define the problem

Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be a Banach space, $\mathrm{BL}(\mathcal{B})$ is the Banach space of all linear bounded operators defined on $\mathcal{B}$ to itself. Its norm is described as follows

$$\forall M \in \mathrm{BL}(\mathcal{B}): \quad \|M\| = \sup_{\|u\|_{\mathcal{B}}=1} \|Mu\|_{\mathcal{B}}.$$

For M, N and K in $\mathrm{BL}(\mathcal{B})$, we define the generalized quadratic spectral problem as follows: Find $\mu \in \mathbb{C}$ and $u \in \mathcal{B}-\{0\}$ such that

$$Q(\mu)u := \mu^2 Mu + \mu Nu + Ku = 0.$$

In the case when M, N and K are matrices, this type of problems is known as the quadratic eigenvalue problem. It was treated by Tisseur and *al*, Huang *al*, Chen and *al* in [5, 6, 7].

Our goal is to generalize the different results obtained in [1, 2, 3, 4] and the studies effected for matrices in [5, 6, 7] .

For that, we define the generalized quadratic resolvent set, noted $\mathrm{re}(M, N, K)$, by

$$\mathrm{re}(M, N, K) = \{\mu \in \mathbb{C} : Q(\mu) \text{ is invertible and bounded}\}$$

and the generalized quadratic spectrum, denoted $\mathrm{sp}(M, N, K)$ by

$$\mathrm{sp}(M, N, K) = \mathbb{C} \setminus \mathrm{re}(M, N, K).$$

In addition, we define the generalized quadratic point spectrum, noted $\mathrm{sp_p}(M, N, K)$ is the set of the generalized quadratic eigenvalues given by:

$$\mathrm{sp_p}(M, N, K) = \{\mu \in \mathbb{C},\ \exists u \in \mathcal{B} \setminus \{0\} : Q(\mu)u = 0\}$$

and the generalized quadratic essential spectrum which is given by the following set:

$$\mathrm{sp}_{\mathrm{ess}}(M,N,K)=\{\mu\in\mathbb{C}: Q(\mu) \text{ is injective, not surjective }\}$$

Then, we can define the generalized quadratic spectrum as follows:

$$\mathrm{sp}(M,N,K)=\mathrm{sp}_{\mathrm{p}}(M,N,K)\cup\mathrm{sp}_{\mathrm{ess}}(M,N,K).$$

Let $\mathrm{R_Q}(\cdot)$ the generalized quadratic solving operator associated to M,N and K define on $\mathrm{re}(M,N,K)\subset\mathbb{C}$ to $\mathrm{BL}(\mathcal{B})$ by

$$\forall z\in\mathrm{re}(M,N,K)\quad \mathrm{R_Q}(z)=Q^{-1}(z)=\left(z^2M+zN+K\right)^{-1}.$$

Theorem 1.1. *if M_n, N_n and K_n converge in collectively compact sense to M,N and K respectively, for all $n\in\mathbb{N}$, $\mu_n\in\mathrm{sp}(M_n,N_n,K_n)$ and $\mu_n\to\mu$ then $\mu\in\mathrm{sp}(M,N,K)$.*

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ADOMIAN DECOMPOSITION METHODS FOR POPULATION BALANCE EQUATIONS

ACHOUR IMANE AND DR. BELLAGOUN ABDELGHANI

Abstract. Particulate processes are modeled by The Population balance equations (PBEs) which are partial integro differential equations.The population balance equations (PBEs) rarely has an analytical solution. However, few cases with assumed functional forms of breakage and aggregation kernels, daughter particle distribution exist. A typical population balance equations include spatial transport terms ,i.e. convection and diffusion terms. In our work we try to solve PBEs for breakage and aggregation with these terms using a powerful technique called Adomian decomposition method. This technique overcomes the crucial difficulties of numerical discretization and stability that often characterize previous solutions in this area. the solutions which we obtained using this technique are analytical and are not available in the literature.

Keywords and phrases. Population Balance model Adomian method-Adomian polynomial

1. Define the problem

The subject consists in developing numerical methods for a certain class of integro-differential equations representing population balance models. The method will be supported by powerful analytical tools such as Adomian-type polynomial procedures.

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ANALYSE OF A LOCAL PROJECTION FINITE ELEMENT STABILIZATION OF NAVIER-STOKES EQUATIONS

JOANNA DIB, DJILALI AMEUR, AND SÉRÉNA DIB

ABSTRACT. We analyze a pressure stabilized finite element method for the approximation of the unsteady Navier-Stokes equations and investigate their stability and convergence properties. We mainly concentrate on the analyse of an equal-order finite element pair for velocity and pressure. We present a particular framework that allows the introduction of a minimal stabilizing term which have better local conservation properties, to overcome the lack of the so-called Ladyzenskaja-Babuska-Brezzi condition and eliminate the inconsistency when equal-order approximations for velocity and pressure are employed. As a result, the stabilized method leads to optimal rates of convergence for both velocity and pressure approximations.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 65N12, 65N30, 65N15, 76D05, 76N10, 35Q30.

KEYWORDS AND PHRASES. Navier-Stokes equations modeling, Stabilized finite element method, local-projection method, Error estimate.

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APPLICATION OF THE GENERALIZED MULTIQUADRIC METHOD FOR SOLVING ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

SELMA BOUZIT AND REBIHA ZEGHDANE

ABSTRACT. In general, the mathematical description of physical processes leads to partial differential equations. In some cases the exact solution can be obtained by using analytical tools or some perturbations methods but in more experimental and practical situations is possible by numerical methods. In this work, we have used generalized multiquadric approximations radial basis functions for solving some elliptic partial differential equations with Dirichlet or Newmann boundary conditions as two dimensional Laplace, and Poisson equations. The subject here is to found a good agreement between exact and numerical solution by using some choices of generalized radial basis functions to obtain excellent approximations.

AMS SUBJECT CLASSIFICATION: 35XX, , 45XX35Q99, 65D05, 65Z99, 34K28.

KEYWORDS: Radial basis functions, Laplace equation, Poisson equation, Boundary conditions, Numerical solution.

1. INTRODUCTION

In 1968, Hardy introduced the multiquadric (MQ) method for the construction approximate two dimensional surfaces of field data. The importance of multiquadric is recongnized by its successful in some application in economics, geography...etc, based on some comparaison scattered data shemes, Franke has concluded that MQ performs the best in accuracy against difference element methods. After Kansa [1] used this multiquadric radial basis functions for the solution of PDEs in computational fluide dynamics. Several papers have been written to show the excellent performance of this radial basis function which replace element and difference methods and it has exponential convergence. In this work, we use the generalized radial basis function to solve some elliptic partial differential equation as test examples, we used the method of Cross validation to estimate the optimal shape parameter. The problem yields to a system of algebraic equations which can be solved for the unknoun coefficients. In the following, we present examples for determining solutions of elliptic PDE using generalized multiquadrics radial basis functions.
As a numerical example we consider first two dimensional Poisson problem

$$U_{xx} + U_{yy} = f(x, y), \tag{1}$$

on a unit cercle domain Ω.
The function f in equation (1) is specified for that the exact solution is

1

$$U(x,y) = \frac{65}{65 + (x - 0.2)^2 + (y - 0.1)^2}.$$

Dirichlet boundary conditions are given on the boundary $\partial\Omega$
The centers of radial basis functions are determined by an optimal algorithm and the errors in this approximation are given over a range of the shape parameter.
Secondly, in addition of Dirichlet boundary conditions, Newmann type boundary conditions as well as mixed type. We consider the Poisson problem on the unit square, the function f is set to

$$f(x,y) = -2(2y^3 - 3y^2 + 1) + 6(1 - x^2)(2y - 1),$$

and Dirichlet boundary conditions of $U(0,y) = 2y^3 - 3y^2 + 1$ and $U(1,y) = 0$ are applied as Newmann boundary conditions of $\frac{\partial U}{\partial y} = 0$ along $y = 0$ and $y = 1$. The second example is the $1d$ nonlinear boundary value problem

$$U_{xx} + U_x - U = f,$$

on the interval $[0,1]$, the function f is specified from that

$$U(x) = x^2 \exp x,$$

is the exact solution. Dirichlet boundary conditions $U(0) = 0$ and $U(1) = e$ are applied.
In this example, we use some technique of linearization to solve this problem.
The third problem is the following Poisson equation

$$U_{xx} + U_{yy} = (\lambda^2 + \eta^2) \exp(\lambda x + \eta y), \quad (x,y) \in \Omega,$$

with

$$U(x,y) = \exp(\lambda x + \eta y), \quad (x,y) \in \partial\Omega,$$

is considered as an example with different values of λ and η
The exact solution is

$$U(x,y) = \exp(\lambda x + \eta y).$$

The domain Ω is taken to be a quarter of circle. All the examples are solved by the generalized RBF with different values of the exponent and compared with the results of MQ RBF interpolation. We conclude from the test equation that the method is shown to work well for the Laplace and Poisson elliptic equations using sufficiently large number of terms to acheive absolute errors of order 10^{-5}. All test equtions are also solved by Hardy's MQ radial basis functions for comparaison. The numerical results shown that the generalized radial basis gives a good results for solving elliptic PDEs.

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ADAPTIVE MODIFIED PROJECTIVE SYNCHRONIZATION OF DIFFERENT FRACTIONAL-ORDER CHAOTIC SYSTEMS WITH UNKNOWN PARAMETERS

HADJER ZERIMECHE, TAREK HOUMOR, AND ABDELHAK BERKANE

ABSTRACT. This work focus on the adaptive modified projective synchronization (AMPS) method to synchronize two different fractional-order chaotic systems (FOCS) with uncertain parameters. The aim of (AMPS) is to guarantee the synchronization of (FOCS) by using the Lyapunov stability theory, an adaptive controller and some techniques of fractional calculus. We use the (AMPS) method to show how the (FOCS) can be synchronized by driving its output to the desired pattern. The important feature of (AMPS) method is the synchronization between almost all (FOCS) with known or unknown parameters can achieve.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 34A08, 34D06.

KEYWORDS AND PHRASES. Synchronization, adaptive control, fractional-order chaotic systems.

1. MAIN RESULTS

Consider the fractional-order drive and response systems with uncertain parameters, respectively, as follows:

$$ {}^{c}D_t^q x = f(x) + F(x)\alpha, \tag{1}$$

$$ {}^{c}D_t^q y = g(y) + G(y)\beta + u. \tag{2}$$

Where $0 < q < 1$ are the fractional-orders, $f(x), g(y) \in \mathbb{R}^n$, are vector functions;
$F(x) \in \mathbb{R}^{n\times m}, G(y) \in \mathbb{R}^{n\times p}$, are matrix functions; $\alpha \in \mathbb{R}^m, \beta \in \mathbb{R}^p$ are uncertain parameter vectors.
Our goal is to design (AMPS) between the system (1) and the system (2) by constructing an effective adaptive controller.
In this paper, the synchronization error between the drive and response systems is defined by $e = x - \theta y$, where θ is diagonal matrix which called scaling factor matrix $\theta = diag(\theta_{11}, \theta_{22}, \ldots, \theta_{nn})$, $\theta_{ii} \neq 0$, $(i = 1 \ldots n)$.
Then

$$\begin{aligned} {}^{c}D_t^q e &= {}^{c}D_t^q x - \theta\, {}^{c}D_t^q y \\ &= f(x) + F(x)\alpha - \theta g(y) - \theta G(y)\beta - \theta u. \end{aligned} \tag{3}$$

Theorem 1.1. *The controller u is proposed as the following*

$$ u = \theta^{-1} f(x) + \theta^{-1} F(x)\tilde{\alpha} - g(y) - G(y)\tilde{\beta} + \theta^{-1} k e, \tag{4}$$

and adaptive law of parameter is taken as

$$\begin{cases} {}^cD_t^q\tilde{\alpha} = [F(x)]^T e + \varepsilon(\alpha - \tilde{\alpha}), \\ {}^cD_t^q\tilde{\beta} = [-G(y)]^T \theta e + \eta(\beta - \tilde{\beta}). \end{cases} \tag{5}$$

Then, the (AMPS) between drive system (1) and response system (2) can be achieved by using the controller (4) and parameter updating law (5).

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AN IMPROVING PROCEDURE OF THE INTERIOR PROJECTIVE ALGORITHM FOR LINEAR SEMIDEFINIT OPTIMIZATION PROBLEMS

EL AMIR DJEFFAL, MOUNIA LAOUAR, AND MAHMOUD BRAHIMI

ABSTRACT. In this paper, a practical modification of Karmarkars projective algorithm for linear semidefinit optimization programming problems. This modification leads to a considerable reduction of the cost and the number of iterations.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 90C05, 90A51.

KEYWORDS AND PHRASES. Linear semidefinit programming; Interior point methods; Karmarkars algorithm; YeLustig algorithm

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AN ITERATIVE REGULARIZATION METHOD FOR AN ABSTRACT ILL-POSED BIPARABOLIC PROBLEM

ABDELGHANI LAKHDARI AND NADJIB BOUSSETILA

Abstract. In this work, we are concerned with the problem of approximating a solution of an ill-posed biparabolic problem in the abstract setting. In order to overcome theinstability of the original problem, we propose a regularizing strategy based on the Kozlov-Maz'ya iteration method. Finally, some other convergence results including some explicit convergence rates are also established under a priori bound assumptions on the exact solution.

2010 Mathematics Subject Classification. 47A52, 65J22.

Keywords and phrases. inverse problem, biparabolic problem, iterative regularization.

1. Formulation of the problem

We consider the inverse source problem of determining the unknown source term $u(0) = f$ and the temperature distribution $u(t)$ for $0 \leq t < T$, in the following biparabolic problem:

$$\begin{cases} \left(\frac{d}{dt} + A\right)^2 u(t) = u"(t) + 2Au'(t) + A^2 u(t) = 0 & 0 < t < T \\ u(T) = g \quad u_t(0) = 0 & , \end{cases}$$

where A is a positive, self-adjoint operator with compact resolvent.

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ANALYSIS AND OPTIMAL CONTROL OF A MATHEMATICAL MODELING OF THE SPREAD OF AFRICAN SWINE FEVER VIRUS WITH A CASE STUDY OF SOUTH KOREA AND COST-EFFECTIVENESS

ABDELFATAH KOUIDERE, OMAR BALATIF, AND MOSTAFA RACHIK

Abstract. In this work, we study a mathematical model describing the dynamics of the transmission of African Swine Fever Virus (ASFV) between pigs on the one hand and ticks on the other hand. The aim is to Protecting pigs against the African swine fever virus. We analysis the mathematical model by using Routh–Hurwitz criteria, the local stability of ASFV-free equilibrium and ASFV equilibrium are obtained. We also study the sensitivity analysis of the model parameters to know the parameters that have a high impact on the reproduction number R_0. The aims of this paper is to reduce the number of infected pigs and ticks. By proposing several strategies, including the iron fencing to isolate uninfected pigs, spraying pesticides to fight ticks that transmit the virus, and getting rid of the infected and suspected pigs. Pontryagin's maximal principle is used to describe the optimal controls and the optimal system is resolved in an iterative manner. Numerical simulations are performed using Matlab. The increased cost-effectiveness ratio was computed to investigate the cost effectiveness of all possible combinations of the three controls measures. Using a cost-effectiveness analysis, we showed that controlling the protection of susceptible pigs, to prevent contact between infected pigs and infected ticks on one hand and susceptible pigs on the other hand, it is the most cost-effective strategy for disease control.

2010 Mathematics Subject Classification. xxxx, xxxx, xxxx.

Keywords and phrases. African swine fever virus, Optimal control, Local stability, Mathematical model, ASF virus.

1. Define the problem

We consider a mathematical model $S_PI_PR_PS_TI_T$, that describes the transmission of African swine fever virus in pigs population. We divide the population denoted by N into five compartments: pigs susceptible S_P, the pigs infected I_P, the pigs recovered R, ticks susceptible S_P and the ticks infected I_T. Hence, the dynamics of the spread of African swine fever virus mathematical model is governed by the following system of differential equation:

$$
(1) \qquad \begin{cases}
\frac{dS_P(t)}{dt} = \Lambda_P - \mu_P \; S_P(t) - \beta_1 \frac{S_P(t)I_P(t)}{N} - \beta_2 \frac{S_P(t)I_T(t)}{N} \\
\frac{dI_P(t)}{dt} = \beta_1 \frac{S_P(t)I_P(t)}{N} + \beta_2 \frac{S_P(t)I_T(t)}{N} - (\; \mu_P + \alpha + \delta)I_P(t) \\
\frac{dR_P(t)}{dt} = \alpha I_P(t) - \mu_P R_P(t) \\
\frac{dS_T(t)}{dt} = \Lambda_T - \mu_T \; S_T(t) - \beta_3 \frac{S_T(t)I_P(t)}{N} \\
\frac{dI_T(t)}{dt} = \beta_3 \frac{S_T(t)I_P(t)}{N} - \mu_T \; I_T(t)
\end{cases}
$$

where $S_P(0) \geq 0$, $I_P(0) \geq 0$, $R_P(0) \geq 0$, $S_T(0) \geq 0$ and $I_R(0) \geq 0$ are the initial state.

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ANALYSIS OF A ELECTRO-VISCOELASTIC CONTACT PROBLEM WITH WEAR AND DAMAGE

ABDELAZIZ AZEB AHMED

Abstract. We study a quasistatic problem describing the contact with friction and wear between a piezoelectric body and a moving foundation. The material is modeled by an electro-viscoelastic constitutive law with long memory and damage. The wear of the contact surface due to friction is taken into account and is described by the differential Archard condition. The contact is modeled with the normal compliance condition and the associated law of dry friction. We present a variational formulation of the problem and establish, under a smallness assumption on the data, the existence and uniqueness of the weak solution. The proof is based on arguments of parabolic evolutionary inequations, elliptic variational inequalities and Banach fixed point.

2010 Mathematics Subject Classification.35J85, 49J40, 47J20, 74M1.

Keywords and phrases. Quasistatic process, electro-viscoelastic materials, damage, normal compliance, friction, wear, existence and uniqueness, fixed point arguments, weak solution.

1. Problem P

Find a displacement field $\boldsymbol{u} : \Omega\times[0,T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega\times[0,T] \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times [0,T] \to \mathbb{R}$ *, an electric displacement field* $\boldsymbol{D} : \Omega \times [0,T] \to \mathbb{R}^d$, a damage field $\beta : \Omega \times [0,T] \to \mathbb{R}$ and a wear

1

function $\zeta : \Gamma_3 \times [0,T] \to \mathbb{R}$ such that

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \beta) + \int_0^t M(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \tag{1}$$

$$+\mathcal{E}^*\nabla\varphi \quad \text{in } \Omega \times (0,T),$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{B}\nabla\varphi \ \text{ in } \Omega \times (0,T), \tag{2}$$

$$\dot{\beta} - k_e \triangle \beta + \partial\Psi_K(\beta) \ni S(\boldsymbol{\varepsilon}(\boldsymbol{u}), \beta) \ \text{ in } \Omega \times (0,T), \tag{3}$$

$$Div\ \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega \times (0,T), \tag{4}$$

$$div\boldsymbol{D} = q_0 \text{ in } \Omega \times (0,T), \tag{5}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{6}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{7}$$

$$\begin{cases} -\sigma_\nu = p_\nu, \\ |\boldsymbol{\sigma}_\tau| \le \mu p_\nu, \\ \boldsymbol{\sigma}_\tau = -\mu p_\nu \frac{(\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*)}{|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*|} \quad \text{if } \dot{\boldsymbol{u}}_\tau \neq \boldsymbol{v}^*, \\ \dot{\zeta} = k_1 \mu\ p_\nu\, R^*(|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*|), \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{8}$$

$$\frac{\partial\beta}{\partial\boldsymbol{\nu}} = 0 \ \text{ on } \Gamma \times (0,T), \tag{9}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{10}$$

$$\boldsymbol{D}\cdot\boldsymbol{\nu} = q_2 \quad \text{on } \Gamma_b \times (0,T), \tag{11}$$

$$\boldsymbol{D}\cdot\boldsymbol{\nu} = \psi(u_\nu - h - \zeta)\phi_L(\varphi - \varphi_0) \quad \text{on } \Gamma_3 \times (0,T), \tag{12}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \beta(0) = \beta_0,\ \zeta(0) = 0 \ \text{ in } \Omega. \tag{13}$$

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ANALYSIS OF FRACTIONAL NONLINEAR OSCILLATORS

TEFAHA LEJDEL ALI, SAFIA MEFTAH, AND LAMINE NISSE

ABSTRACT. In this work a modified version of the forced Van der Pol oscillator in their general form is proposed, introducing fractional-order time derivatives into the state-space model The resulting fractional-order Van der Pol oscillator is analyzed in the time and frequency domains, using phase portraits, spectral analysis and bifurcation diagrams. The fractional-order dynamics is illustrated through numerical simulations of the proposed schemes using approximations to fractional-order operators.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 65-xx, 65Cxx, 65C20

KEYWORDS AND PHRASES. The forced Van der Pol oscillator, nonlinear oscillator, Fractional order operators..

1. DEFINE THE PROBLEM

$$x^{(1+\lambda)} + \epsilon(ax^2 + b(x^{(\lambda)})^2 + cx + d)(x^{(\lambda)})^n = f(x, x^{(\lambda)}, t, \lambda)$$

Where $a, b, c, d \in \mathbb{R}, n \in \mathbb{N}, 0 < \lambda < 1$.

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ANALYSIS OF MATHEMATICAL MODEL OF PROSTATE CANCER WITH ANDROGEN DEPRIVATION THERAPY

ASSIA ZAZOUA AND WENDI WANG

Abstract. A stochastic model of prostate cancer under continuous androgen suppression therapy in [1] is investigated to show the effects of noises, different competition intensities and dosage amount on treatment strategy. Threshold conditions between extinction and persistence in mean for the stochastic system are obtained where noises play an important role in persistence and eradication of tumor cells. Sufficient conditions for the existence of an ergodic stationary distribution are established. Furthermore, the optimal treatment is approximated by using numerical simulations. The analysis of this model suggests that a medicament that overlap with the replication of tumor cells is advantageous for treatment with CAS therapy because of the similar effect of noise disturbances. In addition, the results motivate physicians to find a drug that would reduce the activity of resistance cells in order to prevent relapse and reduce the severity of cancer if it can not be cured.

2010 Mathematics Subject Classification. 76M35,60H30, 65Cxx.

Keywords and phrases. Stochastic noise; Resistance; Persistence; Extinction; Stationary distribution

1. Define the problem

Mathematics is used in oncology since cancer is one of the most deadly diseases in recent years. In fact, on the basis of model [2] and model [8] we have formulated a stochastic competition model of prostate cancer under continuous androgen suppression therapy given by

$$\begin{cases} dA &= [-\gamma(A-a_0)-\gamma a_0 u]dt, \\ dX_1(t) &= \left\{ r_1 A\left(1-\dfrac{X_1+\alpha X_2}{K}\right)-(d_1+m_1)(1-\frac{A}{a_0})\right\} X_1 dt \\ &+ \sigma_1 X_1 dB_1(t), \\ dX_2(t) &= \left\{ r_2\left(1-\dfrac{\beta X_1+X_2}{K}\right)X_2+m_1(1-\frac{A}{a_0})X_1\right\} dt+\sigma_2 X_2 dB_2(t), \end{cases}$$

where, X_1 and X_2 are the concentrations of androgen-dependent cells and androgen-independent cells respectively, A is the concentration of androgen in the blood. All the parameters in the system are positive. $B_i(t)$, $i=1,2$ are independent Brownian motions; σ_1 and σ_2 denote the intensities of the white noises, respectively.

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ANALYTICAL SOLUTION OF TWO DIMENSIONAL FLOW UNDER A GATE USING THE HODOGRAPH METHOD

MAY MANAL BOUNIF AND ABELKADER GASMI

Abstract. The problem of steady two-dimensional free-surface flow of a fluid issued under a gate is considered. The hodograph method is used to solve this problem analytically. The principle idea of this method is based on the transformation of the domain occupied by the fluid in the physical plane on to a part of unit disk. This simplifies the problem such that it becomes one-dimensional instead of two-dimensional. The obtained result are agree with the numerical results given by Gasmi & Mekias [3]

2010 Mathematics Subject Classification. 76Bxx, 76B07.

Keywords and phrases.Free-surface, zero gravity, hodograph transformation, incompressible flow.

1. Define the problem

The steady two-dimensional irrotational flow issuing under a gate in the absence of the surface tension forces (see Figure 1) is considered. The fluid is inviscid and incompressible. The mathematical problem is to seek to the

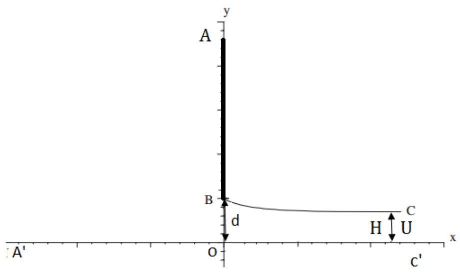

Figure 1. Sketch of the flow and the coordinates

velocity potential functionφ who satisfy the governing equations of the flow:

$$\Delta\varphi = 0, \qquad \text{in the flow field.} \tag{1}$$

$$\frac{\partial\varphi}{\partial\eta} = 0, \qquad \text{on the walls} \tag{2}$$

where η is a normal vector of the boundaries.

$$\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial\varphi}{\partial y}\right)^2 = cst, \qquad \text{on the free surface.} \tag{3}$$

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$$\varphi \longrightarrow cst, \qquad x \longrightarrow -\infty \tag{4}$$

$$\varphi \longrightarrow Ux, \qquad x \longrightarrow +\infty \tag{5}$$

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APPLICATION OF METAHEURISTICS IN SOLVING INITIAL VALUE PROBLEMS (IVPS).

OUAAR FATIMA

Abstract. Some differential equations admit analytic solutions given by explicit formulas. However, in most other case only approximated solutions can be found. Several methods are available in the literature to find approximate solutions to differential equations. Numerical methods form an important part of solving IVP in ODE, most especially in cases where there is no closed form of solutions. The present paper focus the attention toward solving IVP by transforming it to an optimization approach which can be solved through the application of non-standard methods called Metaheuristic. By transforming the IVP into an optimization problem, an objective function, which comprises both the IVP and initial conditions, is constructed and its optimum solutions represents an approximative solution of the IVP. The main contribution of the present paper is divided in twofold. In the one hand, we consider IVPs as an optimization problem when the search of the optimum solution is performed by means of MAs including ABC, BA and FPA and a set of numerical methods including Euler methods, Runge–Kutta methods and predictor–corrector methods. On the other hand, we propose a new MA called Fractional Lévy Flight Bat Algorithm (FLFBA) (which is an improvement of the BA, based on velocity update through fractional calculus and local search procedure based on a L´evy distribution random walk). We illustrates its computational efficiency by comparing its performance with the previous methodds in solving the bacterial population growth models (both the logistic growth model and the exponential growth model).

2010 Mathematics Subject Classification. 65L05, 90C59.

Keywords and phrases. Initial Value Problem (IVP), Optimization problem, Exponential problem, Logistic problem, FLFBA, Numerical methods, Metaheuristic algorithms.

1. Define the problem

To illustrate the FLFBA's performance and to demonstrate its computationally efficiency, we select - as a studied problem - the bacterial population growth models that are the logistic growth and the exponential growth models by taking a uniform step size h.

The main motivation in the selection of the application examples comes from the great importance of the exponential equation in modeling any phenomena where a quantity is allowed to undergo unrestrained growth, while the logistic differential equations [8] are an ODE whose solution is a logistic function, they are useful in various other fields as well, as they often provide significantly more practical models than exponential ones which fail to take into account constraints that prevent indefinite growth, and logistic functions correct this error. They are also useful in a variety of other contexts, including machine learning, chess ratings, cancer treatment (i.e. modeling tumor growth), economics, and even in studying language adoption. The logistic function is shown to be the solution of the Riccati equation, some second-order nonlinear ODEs and many third-order nonlinear ODEs.

In this paper, the IVP is formulated as an optimization problem [3, 4, 5, 6, 7] it will be solved with FLFBA compared to several methods including Euler's methods (Explicit Euler, Midpoint method and Backward Eulers), Runge–Kutta methods (RK4, Heuns (RK2)) and predictor–corrector methods (Adams–Bashforth–Moulton method (ABM)). Then FLFBA is compared with three MAs that are: Artificial Bee Colony Algorithm (ABCA) inspired by the behavior of honey bees, Bat Algorithm (BA) simulates the echolocation behavior of bats and Flower Pollination Algorithm (FPA) inspired by the flower pollination process of flowering plants [9] to examine which algorithm find the best numerical solutions with the best effectiveness for the studied problem. All computations were performed on an MSWindow 2007 professional operating system in the Matlab environment version R2013a compiler on Intel Duo Core 2.20 GHz. PC.

2. Problem Formulation

We consider the general Cauchy problem as:

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}, \tag{1}$$

where t is the independent variable and $y = y(t)$ is the dependent variable. By using the classical assumption:

$$f : [t_0 - T, t_0 + T] \times [y_0 - Y, y_0 + Y] \to \mathbb{R},$$

is continuous and satisfies the Lipschitz condition:

$$|f(t, y_1) - f(t, y_2)| \leq L\,|y_1 - y_2|\,,$$

it results there exists a single solution y. There are many methods used to find the solution, but, in practice, we always solve the problem by using numerical methods, like Runge-Kutta or Euler methods but these classical mathematical tools are not very precise. The main goal of this thesis is to underline the possibility of using a different method, based on metaheuristic algorithms like FLFBA.

2.1. **Objective Function.** Finding the values of the unknown function $y = y(t), y : [a, b] \to \mathbb{R}$, according to a finite set of equidistant values of the independent variable $t_0 = a < t_1 < ... < t_n = b, t_i = a + ih, h = \dfrac{b-a}{n}$. We denote by $y_i = y\,(t_i)\,, i = 1...n$ the values of the unknown function y, in accordance with the given division. Thus, the vector $(y_1, y_2, ..., y_n)$ is an admissible solution. We will consider the population as being a subset of admissible solutions. Given an instant t, we denote the population by $Y(t)$. One individual $y = (y_1, y_2, ..., y_n)$ is characterized by the values y_i. The individuals in a natural population are, more or less, adapted. Thus, in order to simulate natural selection, we will select, in each stage, only one subset of individuals, namely those who are best adapted. The surplus of individuals is eliminated, taking into account the decreasing values of the objective function. In order to evaluate each individual, we will use the following approximate formula (finite difference formula) for the derivative:

$$\acute{y}(t_i) \approx \frac{y_i - y_{i-1}}{h}, \qquad \left|\acute{y}(t_i) - \frac{y_i - y_{i-1}}{h}\right| \leq const.h.$$

Consequently, the discrete form of the Cauchy problem will be:

$$\frac{y_i - y_{i-1}}{h} = f(t_i, y_i), \qquad i = 1...n. \tag{2}$$

The above system is, generally, nonlinear. Finding the vector $(y_1, y_2, ..., y_n)$ which satisfies the above conditions is our goal. Of course, for an admissible solution, we do not have the equality in Eq. (2) and, consequently, we have to consider the error formula:

$$\left(\frac{y_i - y_{i-1}}{h} - f(t_i, y_i)\right)^2.$$

The objective function associated to an individual $y = (y_1, y_2, ..., y_n)$ will be:

$$F(y) = \sum_{i=1}^{n} \left(\frac{y_i - y_{i-1}}{h} - f(t_i, y_i)\right)^2. \tag{3}$$

An individual from $Y(t)$ will be better adapted if it implies a smaller value of the function F. Each individual may suffer some modifications, which may be hazardous, we will consider that $y_i \pm \varepsilon$ is a mutation for y_i.

2.2. **Consistency.** We denote by u_t the best adapted individual in the population, at the instance t, i.e. the individual in the population which has the minimum value of the function F. In [2] it's already stated that the sequence $(u_t)_{t \geq 0}$ converges, its limit being the solution of the optimization problem $\inf F$. While the solution is the limit of a convergent sequence, by applying the optimization algorithms, the following assertion is true:

For $\varepsilon > 0$, there is a $(y_1, y_2, ..., y_n)$ such that:

$$F(y) = \sum_{i=1}^{n} \left(\frac{y_i - y_{i-1}}{h} - f(t_i, y_i)\right)^2 < \varepsilon,$$

it results there is a $y = (y_1, y_2, ..., y_n)$ such that:

$$\left|\frac{y_i - y_{i-1}}{h} - f(t_{i,} y_i)\right| < h,$$

taking into account the approximation of the derivative, we have:

$$|\acute{y}(t_i) - f(t_{i,} y_i)| \leq \left|\acute{y}(t_i) - \frac{y_i - y_{i-1}}{h}\right| + \left|\frac{y_i - y_{i-1}}{h} - f(t_{i,} y_i)\right| < Ch,$$

when C denotes a positive constant. The last relation shows that the final value $y = (y_1, y_2, ..., y_n)$ is an approximate solution of the Cauchy problem, for small values of h.

3. Population Growth Models

In this section we explain briefly the two population growth models:

3.1. **Exponential growth.** Suppose that $P(t)$ describes the quantity of a population at time t. For example, $P(t)$ could be the number of milligrams of bacteria in a particular beaker for a biology experiment at a time t. A model of population growth tells plausible rules for how such a population changes over time. The simplest model of population growth is the exponential model, which assumes that there is a constant parameter r, called the growth parameter, such that:

$$\acute{P}(t) = rP(t),$$

holds for all time t. This differential equation it self might be called the exponential differential equation, because its solution is:

$$P(t) = P_0 e^{rt}, \tag{4}$$

where $P_0 = P(0)$ is the initial population. One noticeable feature of the exponential model is that, when r is positive, the population always grows larger and larger without any finite limit. This kind of growth may be a good model for a new population of bacteria in a beaker, but it does not hold for a long time. It is easy to see that the equation would imply a population of bacteria that ultimately outgrew the beaker and even outgrew the planet earth, since the mass of the bacteria would ultimately exceed the mass of the earth. Such a model is therefore absurd to model a system for long periods of time. The fundamental difficulty is that the exponential differential equation ignores the fact that there are limits to resources needed for the population to grow. It ignores the needs for food, oxygen, and space; and it ignores the accumulation of waste products that inevitably arise. The logistic curve gives a much better general formula for population growth over a long period of time than does exponential growth.

3.2. **Logistic growth.** An alternative model was proposed by Verhulst in 1836 [8] to allow for the fact that there are limits to growth in all known biological systems. He proposed what is now called the logistic differential equation. The equation involves two positive parameters. The first parameter r is again called the growth parameter and plays a role similar to that of r in the exponential differential equation. The second parameter K is called the carrying capacity. The logistic differential equation is written:

$$\acute{P}(t) = rP(t)[\frac{K - P(t)}{K}].$$

Equivalently, in terms of the d notation, the logistic differential equation is:

$$\frac{dP}{dt} = rP[\frac{K - P(t)}{K}].$$

Note that when $P(t)$ is very small, then $P(t)/K$ is close to 0, so the entire factor $[\frac{K-P(t)}{K}]$ is close to 1 and the equation itself is then close to $\acute{P}(t) = rP(t)$; we then expect that the population grows approximately at an exponential rate when the population is small. On the other hand, if $P(t)$ gets to be near K, then $P(t)/K$ will be approximately 1, so $[\frac{K-P(t)}{K}]$ will be approximately 0, and the logistic differential equation will then say approximately $\acute{P}(t) = rP(t)0 = 0$. The growth rate will be essentially 0, so the population will not grow significantly more. The solution of the logistic differential equation is:

$$P(t) = \frac{P_0 K}{P_0 + (K - P_0)e^{-rt}}, \tag{5}$$

where $P_0 = P(0)$ is the initial population. This formula is the logistic formula. It tells the equation for the logistic curve.

4. Numerical Experiments

The implementation of any numerical method could turn difficult because it is necessary to take into account several issues as the discretization order, the algorithm stability, the convergence speed, how to fulfill the boundary conditions, etc. In the methods described in this thesis, the original problem is transformed into an optimization one according to Eq. (3). For making a quantitative comparison, this section is devoted to compare the FLFBA with other algorithms, such as other metaheuristic approaches or more traditional numerical methods, two ordinary differential equations (linear and nonlinear IVP) have been solved with traditional numerical methods (see Table 3) and metaheuristic algorithms (see Table 6).

4.1. **Application example.** Consider a bacterial population growth problem, when the initial population is 3 *milligrams* (*mg*) of bacteria, the carrying capacity is $K = 100$ *mg*, and the growth parameter is $r = 0.2$ $hour^{-1}$. We want to find the solutions of the differential equations satisfied by this population by means of FLFBA, ABCA, BA, FPA and more traditional numerical methods and comparing between their performances.

Exponential growth model The exponential growth model is considered as a linear first order IVP, hence based on Eq. (4) the exponential differential equation is given by

$$P(t) = 3e^{0.2t}.$$

Logistic growth model The logistic differential equation related to our example is considered as a Bernoulli differential equation (and also a separable nonlinear first order IVP), solving it using either approach gives the solution as in Eq. (5)

$$P(t) = \frac{(3)(100)}{3 + (100-3)e^{-0.2t}} = \frac{300}{3 + 97e^{-0.2t}}$$

.

4.2. **Parameters adopted to solve IVP.** FLFBA, ABCA, BA and FPA are an optimization instrument. Then, the essential differential equation is converting into discretization form Eq. (3). The difference formula is used to convert differential equation into discretizations form when the derivative term is replaced in the discretized form by a difference quotient for approximations. The IVP related parameters are as follows:

(1) The number of interior nodes ($n = 9$).
(2) The initial condition in our examples is considered by 3 *milligrams* (*mg*) of bacteria and the interval between which the differential equation is $t \in [0, 50]$.

Parameters	Value
Dimension of the search variables (dim)	10
Maximal number of generations (iterations) (M)	100
Population size (pop)	30
The maximal and minimal pulse rate $(r0Max, r0Min)$	$(1, 0)$
The maximal and minimal frequency $(freqDMax, freqDMin)$	$(2, 0)$
The maximal and minimal loudness $(AMax, AMin)$	$(2, 0)$
gamma	0.9
alpha	0.99
The maximal and minimal inertia weight $(wMax, wMin)$	$(0.9, 0.2)$

Table 1. Parameters adopted to generate FLFBA.

Parameters	BA	FBA	ABCA
Dimension of the search variables (d)	10	10	10
Number of generations (N)	100	100	100
Population size (n)	30	30	30
Loudness (constant or decreasing) (A)	0.5	/	/
Pulse rate (constant or decreasing) (r)	0.5	/	/
Probabibility switch (p)	/	0.8	/

Table 2. Parameters adopted to generate BA, FPA and ABCA.

(3) The interval of the IVP is equally partitioned into $(n+1)$ equidistant subintervals with the length $h = (b-a)/n+1$. Since $t \in [0, 50]$ in our example, hence the step size $h = 5$.
(4) The number of generations is set to 100 and the population size is set to 30 for all MAs used in this study.
(5) For a better analysis of the results, a Monte Carlo simulation is performed (i.e. we run the program several times for the same testing problem) so each optimization procedure was repeated 50 times for all MAs and in all dimensions.
(6) The objective function:

$$F(y_1, y_2, ..., y_{10}) = \sum_{j=1}^{10} \left(\frac{y_j - y_{j-1}}{h} - f(t_{j-1}, y_{j-1}) \right)^2 = \sum_{j=1}^{10} \left(\frac{y_j - y_{j-1}}{h} - y_{j-1} \right)^2 .$$

Table (1) indicates the different parameters used to generate FLFBA [1]. Table (2) gives the parameters adopted to generate BA, FPA and ABCA (for more details about these three algorithms see [9].

4.3. **Comparison of FLFBA with numerical methods.** In this subsection, we look into several methods of obtaining the numerical solutions to ordinary differential equations (ODEs) in which all dependent variables (x) depend on a single independent variable (t).

i	x_i	**Exact**	Expl Euler	RK4	Heuns	Midpoint	Back Euler	ABM	**FLFBA**
						Problem 1			
0	0	**3.0000**	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	**3 0000**
1	5	**8.0000**	6.0000	8.0000	8.0000	8.0000	6.0000	8.1250	**8.0000**
2	10	**22.000**	12.000	22.000	19.000	19.000	12.000	22.005	**22 000**
3	15	**60.000**	24.000	60.000	47.000	47.000	24.000	59.597	**60.000**
4	20	**164.00**	48.000	161.00	117.00	117.00	48.000	160.08	**167.00**
5	25	**445.00**	96.000	437.00	293.00	293.00	96.000	429.57	**453.00**
6	30	**1210.0**	192.00	1184.0	732.00	732.00	192.00	1152.8	**1236.0**
7	35	**3290.0**	384.00	3207.0	1831.0	1831.0	384.00	3093.7	**3319.0**
8	40	**8943.0**	768.00	8684.0	4578.0	4578.0	768.00	8302.7	**8977.0**
9	45	**24309**	1536.0	23520	11444	11444	1536.0	22282	**24346**
10	50	**66079**	3072.0	63700	28610	28610	3072.0	59798	**66120**
						Problem 2			
0	0	**3.00000**	3.00000	3.00000	3.00000	3.00000	3.00000	3.00000	**3.00000**
1	5	**7.75510**	5.91000	7.73340	7.23540	7.25650	5.91000	7.73330	**7.75550**
2	10	**18.6017**	11.4707	18.5208	16.5923	16.7499	11.4707	18.5207	**18.6044**
3	15	**38.3174**	21.6257	38.1583	34.0973	34.8446	21.6257	38.1584	**38.3231**
4	20	**62.8060**	38.5746	62.6348	57.6171	59.6999	38.5746	62.5191	**62.8112**
5	25	**82.1112**	62.2692	81.9715	77.1952	79.9782	62.2692	83.0479	**82.1157**
6	30	**92.5800**	85.7639	92.4564	88.4623	90.5498	85.7639	94.1325	**92.5845**
7	35	**97.1360**	97.9733	97.0456	94.2223	95.4540	97.9733	96.7510	**97.1384**
8	40	**98.9270**	99.9589	98.8735	97.1106	97.7738	99.9589	97.3996	**98.9275**
9	45	**99.6026**	100.000	99.5749	98.5553	98.8988	100.0000	99.4568	**99.6024**
10	50	**99.8534**	100.000	99.8402	99.2776	99.4523	100.0000	100.493	**99.6024**

Table 3. Comparison of FLFBA with numerical methods.

The IVPs will be handled with several methods including Euler's methods (Explicit Euler, Midpoint method and Backward Eulers), Runge–Kutta methods (RK4, Heuns (RK2)) and predictor–corrector methods (Adams–Bashforth–Moulton method(ABM)). In Matlab we plot the numerical results together with the (true) analytical solution. The results are depicted in Figure (**??**) and listed in Table (3).

Comparison of exact results with those of numerical methods and FLFBA show that the RK4 method is better than Heun's method and ABM's method, while Euler's method is the worst in terms of accuracy with the same step-size, while the FLFBA approach gives the best solution since it does not depend on the type of differential equation i. e., is based on velocity update through fractional calculus and a local search procedure based on an Lévy distribution random walk. The absolute error of the proposed methods are made in the Table (4)

Starting with the Euler method, since it is easy to understand and simple to program. Even though its low accuracy keeps it from being widely used for solving ODEs, it gives us a clue to the basic concept of numerical

i	x_i	Expl Euler	RK4	Heuns	Midpoint	Back Euler	ABM	**FLFBA**
					Problem 1			
0	0	00.000	00.000	00.000	00.000	00.000	00.000	**00.000**
1	5	02.000	00.000	00.000	00.000	02.000	01.000	**00.000**
2	10	10.000	00.000	03.000	03.000	10.000	00.000	**00.000**
3	15	36.000	00.000	13.000	13.000	36.000	04.000	**00.000**
4	20	116.00	03.000	47.000	47.000	116.00	39.000	**03.000**
5	25	349.00	08.000	152.00	152.00	349.00	154.00	**08.000**
6	30	1018.0	26.000	478.00	478.00	1018.0	572.00	**26.000**
7	35	2906.0	83.000	1459.0	1459.0	2906.0	196.20	**29.000**
8	40	8175.0	259.00	4365.0	4365.0	8175.0	640.30	**34.000**
9	45	22773	789.00	12865	12865	22773	202.71	**37.000**
10	50	63007	2379.0	37469	37469	63007	628.14	**41.000**
					Problem 2			
0	0	00.0000	0.0000	0.0000	0.0000	00.0000	0.0000	**0.0000**
1	5	01.8451	0.0217	0.5197	0.4986	01.8451	0.0218	**0.0004**
2	10	07.1310	0.0809	2.0094	1.8518	07.1310	0.0810	**0.0027**
3	15	16.6917	0.1591	4.2201	3.4728	16.6917	0.1590	**0.0057**
4	20	24.2314	0.1712	5.1889	3.1061	24.2314	0.2869	**0.0052**
5	25	19.8420	0.1397	4.9160	2.1330	19.8420	0.9367	**0.0045**
6	30	06.8161	0.1236	4.1177	2.0302	06.8161	1.5525	**0.0045**
7	35	00.8373	0.0904	2.9137	1.6820	00.8373	0.3850	**0.0024**
8	40	01.0319	0.0535	1.8164	1.1532	01.0319	1.5274	**0.0005**
9	45	00.3974	0.0277	1.0473	0.7038	00.3974	0.1458	**0.0002**
10	50	00.1466	0.0132	0.5758	0.4011	00.1466	0.6396	**0.0002**

TABLE 4. Absolute error between exact solution and different methods.

solution for a differential equation simply and clearly. The error of Heun's method is $O(h^2)$ (proportional to h^2), while the error of Euler's method is $O(h)$. Although Heun's method is a little better than the Euler method, it is still not accurate enough for most real-world problems. The global error of the midpoint method is of order $O(h^2)$. Thus, while more computationally intensive than Euler's method, the midpoint method's error generally decreases faster as $h \to 0$. The fourth-order Runge–Kutta (RK4) method having a truncation error of $O(h^4)$ is one of the most widely used methods for solving differential equations The Adams–Bashforth–Moulton (ABM) scheme needs only two function evaluations (calls) per iteration, while having a truncation error of $O(h^5)$.

From Table (5) that show the maximum error of MATLAB built-in routine "ode 45" compared with different numerical methods and FLFBA approach with step size $h = 5$, we can see that the RK4 method gives a better numerical solution with less error and shorter computation time (see Table (8)) than the MATLAB built-in routine "ode45", as well as the FLFBA (but, a general conclusion should not be deduced just from one example).

	Expl Euler	RK4	Heuns	Midpoint	Back Eulers	ABM	FLFBA
Problem 1	63019.6056	2391.1697	37481.3761	37481.3761	63019.6056	6294.4	33
Problem 2	24.2366	0.17644	5.1942	3.4785	24.2366	1.5480	0.000

Table 5. Maximum error of ode45 vs. difrerent numerical methods with step size h= 5.

4.4. **Comparison of FLFBA with metaheuristic algorithms.** In this subsection, the IVP is formulated as an optimization problem (Eq. 3) solved with three metaheuristics that are: ABCA inspired by the behavior of honey bees, BA simulates the echolocation behavior of bats and FPA inspired by the flower pollination process of flowering plants as well as the FLFBA, by focusing on the performance of these three algorithms compared to FLFBA's performance to examine which one finds the best numerical solutions with the best effectiveness for the studied problems. The obtained results, the comparison of the proposed algorithms to the exact solution are shown in Table (6)

After a comparison between the exact solution and the algorithms outcomes of the chosen examples; the results found that FLFBA is very adequately precise than ABCA, BA and FPA in both exponential and logistic growth models since it possesses the smallest error. The absolute error of the proposed algorithms are made in the Table (7). The comparison between the performances of BA, FPA, ABCA and FLFBA face to the exact results confirm that FLFBA is better because it has a very close curve to the exact curve contrary to the other methods. In both representations of the absolute error (tabular and graphical), FLFBA method offers a very negligible absolute error compared to the other methods.

4.5. **Time taken for the algorithms.** The major factors to be considered in evaluating/comparing different numerical methods is the accuracy of the numerical solution and its computation time. Table (8) shows the time taken for the different studied algorithms. In this comparison, we can say that in some cases the MAs can achieve a more accurate solution using less time consuming than the numerical methods because of in the MAs the solutions obtained are coded in a more compact way requiring significantly less amount of memory.

It is important to note that the evaluation/comparison of numerical methods is not so simple because their performances may depend on the characteristic of the problem at hand. It should also be noted that there are other factors to be considered, such as stability, versatility, proof against runtime errors, and so on.

5. Conclusion

Throughout this chapter, application of standard ABCA, BA, FPA, some numerical methods for solving IVP compared to FLFBA is discussed when they are used as a tool for optimize numerically the IVPs arising in environmental field that is differential equations describing the growth phenomena

i	x_i	Exact	BA	FPA	ABCA	**FLFBA**
				Problem 1		
0	0	03.000	06.000	04.000	11.000	**03.000**
1	5	08.000	15.000	12.000	20.000	**08.000**
2	10	22.000	33.000	29.000	39.000	**22.000**
3	15	60.000	75.000	69.000	83.000	**60.000**
4	20	164.00	183.00	166.00	193.00	**167.00**
5	25	445.00	457.00	452.00	456.00	**453.00**
6	30	1210.0	1238.0	1215.0	1246.0	**1236.0**
7	35	3290.0	3323.0	3319.0	3330.0	**3319.0**
8	40	8943.0	8982.0	8951.0	8988.0	**8977.0**
9	45	24309	24352	24346	24355	**24346**
10	50	66079	61260	66120	66130	**66120**
				Problem 2		
0	0	03.0000	03.0015	03.0010	03.0021	**03.0000**
1	5	07.7551	00.7559	00.7556	00.7561	**07.7555**
2	10	18.6017	18.6063	18.6063	18.6063	**18.6044**
3	15	38.3174	38.3258	38.3249	38.3278	**38.3231**
4	20	62.8060	62.8159	62.8149	62.8179	**62.8112**
5	25	82.1112	82.1228	82.1208	82.1237	**82.1157**
6	30	92.5800	92.5927	92.5909	92.5936	**92.5845**
7	35	97.1360	97.1493	97.1477	97.1501	**97.1384**
8	40	98.9270	98.9412	98.9397	98.9423	**98.9275**
9	45	99.6026	99.6180	99.6160	99.6187	**99.6024**
10	50	99.8534	99.8697	99.8675	99.8708	**99.6024**

TABLE 6. Comparison of FLFBA with MAs.

of such population in both exponential and logistic cases with an initial population via a chosen example.

In the exponential growth problem, results show a population growing always faster without any bond. In reality this model is unrealistic because environments impose limitations to population growth. A more accurate model postulates that the relative growth rate $\frac{\acute{P}}{P}$ decreases when P approaches the carrying capacity K of the environment.

But in the case of logistic growth problem, results show the logistic curve. Note that it has roughly the shape of an elongated S (and it is in fact sometimes called the $S-$ *shaped curve*). The population initially grows slowly but steadily. Then the growth speeds up and the curve moves more steeply upward. As the population gets closer to the carrying capacity $K = 100$, the growth slows and the curve gets more horizontal again. In fact the population never appears to reach the carrying capacity, but instead seems to approach it as an asymptote.

i	x_i	BA	FPA	ABCA	**FLFBA**
			Problem 1		
0	0	03.000	01.000	08.000	**00.000**
1	5	07.000	04.000	12.000	**00.000**
2	10	11.000	07.000	17.000	**00.000**
3	15	15.000	09.000	23.000	**00.000**
4	20	19.000	12.000	29.000	**03.000**
5	25	22.000	17.000	31.000	**08.000**
6	30	28.000	25.000	36.000	**26.000**
7	35	33.000	29.000	40.000	**29.000**
8	40	39.000	35.000	45.000	**34.000**
9	45	43.000	37.000	46.000	**37.000**
10	50	47.000	41.000	51.000	**41.000**
			Problem 2		
0	0	0.0015	0.0010	0.0021	**0.0000**
1	5	0.0008	0.0005	0.0010	**0.0004**
2	10	0.0046	0.0046	0.0046	**0.0027**
3	15	0.0084	0.0075	0.0104	**0.0057**
4	20	0.0099	0.0089	0.0119	**0.0052**
5	25	0.0116	0.0096	0.0125	**0.0045**
6	30	0.0127	0.0109	0.0136	**0.0045**
7	35	0.0133	0.0117	0.0141	**0.0024**
8	40	0.0142	0.0127	0.0153	**0.0005**
9	45	0.0154	0.0134	0.0161	**0.0002**
10	50	0.0163	0.0141	0.0174	**0.0002**

TABLE 7. Absolute error between the exact solution and MAs.

Algorithm	Problem 1	Problem 2
Expl Euler	41×10^{-4} s	38×10^{-4} s
Rk4	70×10^{-4} s	21×10^{-4} s
Heuns	57×10^{-4} s	23×10^{-4} s
Midpoint	52×10^{-4} s	24×10^{-4} s
Back Eulers	51×10^{-4} s	23×10^{-4} s
ABM	51×10^{-4} s	23×10^{-4} s
FLFBA	32×10^{-4} s	21×10^{-4} s
FPA	34×10^{-4} s	21×10^{-4} s
BA	34×10^{-4} s	22×10^{-4} s
ABCA	35×10^{-4} s	23×10^{-4} s

TABLE 8. Time taken for the algorithms.

After a comparison between the exact solutions and the algorithms outcomes; FLFBA was found exponentially better than the other methods by giving accurate solutions with smallest amount error.

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APPROACH SOLUTION FOR FRACTIONAL DIFFERENTIAL EQUATION BY CONFORMABLE REDUCED DIFFERENTIAL TRANSFORMATION METHOD

SAAD ABDELKEBIR AND BRAHIM NOUIRI

Abstract. The Reduced Conformable Differential Transformation method (CRDTM) is used to obtain the solution of the Conformable fractional differential equations.
The Conformable derivative has been studied in many works and research, including Roshdi Khalil and Thabet Abdeljawad. See : [1], [2].
The applications, we offer examples of fractional differential equations and find solutions to them by the (CRDTM), where the results obtained in these methods are compared with each other and with exact solutions. See: [3],[4].
Based on graphical representations of exact and approximate solutions. We can know that this method is a little precise and less suspect method.

2010 Mathematics Subject Classification. 34A05, 34M25, 34A08.

Keywords and phrases. Conformable Derivative, Reduced Differential Transform Method (RDTM), Fractional Differential Equation.

1. Define the problem

We use a reduced differential transformation of Conformable fractional differential equations. By this transformation, we find the approximate solutions of the fractional defferential equations and compare them with the exact solutions. We deduce from the examples whether the approximate solutions have a small error or not.

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APPROXIMATE IMPEDANCE OF A NON PLANAR THIN LAYER IN THE FRAMEWORK OF ASYMMETRIC ELASTICITY.

ATHMANE ABDALLAOUI

Abstract. We consider a two-dimensional transmission problem of linear asymmetric elasticity in a domain Ω_- coated by a thin layer Ω_+^δ. Our aim is to model the effect of the thin layer Ω_+^δ on the fixed domain Ω_- by a mechanical impedance boundary condition. For that we use the techniques of asymptotic expansion. We approximate The transmission problem by a mechanical impedance problem set in the fixed domain Ω_-, and we prove an error estimate.

2010 Mathematics Subject Classification. 11T23, 20G40, 94B05.

Keywords and phrases. Linear elasticity, micropolar body, thin layer, impedance operator.

1. Problem setting

Let Ω^δ be a bounded domain of $\mathbb{R}^2$ consisting of two smooth sub-domains: an open bounded subset Ω_- with regular disjoint regular boundaries Σ and Γ_-, an exterieur domain Ω_+^δ with disjoint regular boundaries Σ and Γ_+^δ (See Figure 1).

Ω_- and Ω_+^δ are open smooth bounded domains in $\mathbb{R}^2$

$$\Omega^\delta = \Omega_- \cup \Omega_+^\delta,\ \partial\Omega_- = \Gamma_- \cup \Sigma,\ \partial\Omega_+^\delta = \Sigma \cup \Gamma_+^\delta$$

The thickness δ of the thin layer Ω_+^δ is supposed to be small enough.

n : is the unit normal vector to Σ outer for Ω_- and inner for Ω_+^δ.

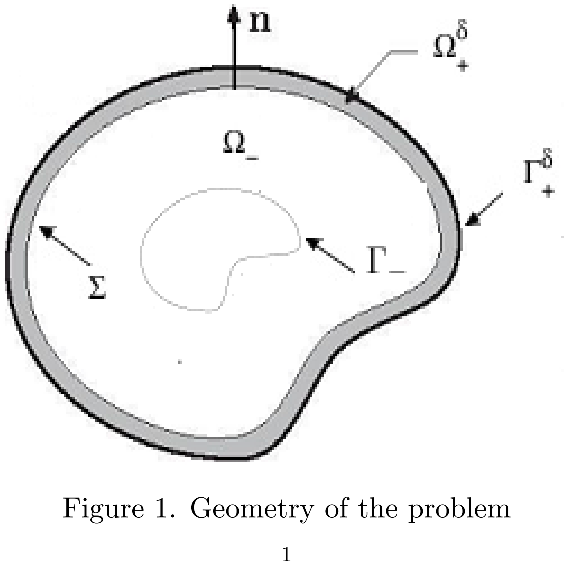

Figure 1. Geometry of the problem

We will interest in the following transmission problem (P^{δ}) for asymmetric linear elasticity (see [1], [7]):

- Equations in Ω_-

$$\left\{\begin{array}{l}\operatorname{div}\left[\sigma_-\left(u_-^{\delta},\omega_-^{\delta}\right)\right]^T=f_-,\\ \left(\nu_-+\epsilon_-\right)\Delta\omega_-^{\delta}+\sigma_{-12}\left(u_-^{\delta},\omega_-^{\delta}\right)-\sigma_{-21}\left(u_-^{\delta},\omega_-^{\delta}\right)=g_-.\end{array}\right. \tag{1}$$

- Equations in Ω_+^{δ}

$$\left\{\begin{array}{l}\operatorname{div}\left[\sigma_+\left(u_+^{\delta},\omega_+^{\delta}\right)\right]^T=0,\\ \left(\nu_++\epsilon_+\right)\Delta\omega_+^{\delta}+\sigma_{+12}\left(u_+^{\delta},\omega_+^{\delta}\right)-\sigma_{+21}\left(u_+^{\delta},\omega_+^{\delta}\right)=0.\end{array}\right. \tag{2}$$

- Boundary conditions on Γ_+^{δ}

$$\left\{\begin{array}{l}\left[\sigma_+\left(u_+^{\delta},\omega_+^{\delta}\right)\right]^T n=0,\\ \left(\nu_++\epsilon_+\right)\frac{\partial\omega_+^{\delta}}{\partial n}=0,\end{array}\right. \tag{3}$$

- Boundary conditions on Γ_-

$$\left\{\begin{array}{l}u_-^{\delta}=0,\\ \omega_-^{\delta}=0.\end{array}\right. \tag{4}$$

- Transmission conditions on Σ

$$\left\{\begin{array}{c}u_-^{\delta}=u_+^{\delta},\\ \omega_-^{\delta}=\omega_+^{\delta},\\ \left[\sigma_-\left(u_-^{\delta},\omega_-^{\delta}\right)\right]^T n=\left[\sigma_+\left(u_+^{\delta},\omega_+^{\delta}\right)\right]^T n,\\ \left(\nu_-+\epsilon_-\right)\frac{\partial\omega_-^{\delta}}{\partial n}=\left(\nu_++\epsilon_+\right)\frac{\partial\omega_+^{\delta}}{\partial n}.\end{array}\right. \tag{5}$$

where $\nu_{\pm}$ and $\epsilon_{\pm}$ are positive material constants, f_- and g_- are body force and body moment, respectively, $u_{\pm}^{\delta}$ and $\omega_{\pm}^{\delta}$ are displacement and rotation fields, respectively

$$\left(u_{\pm}^{\delta},\omega_{\pm}^{\delta}\right)=\left(\left(u_{\pm1}^{\delta},\ u_{\pm2}^{\delta}\right),\ \omega_{\pm}^{\delta}\right)=\left(u_{\pm1}^{\delta},\ u_{\pm2}^{\delta},\ \omega_{\pm}^{\delta}\right),$$

$\gamma_{\pm ji}$ is the asymmetric strain tensor defined by:

$$\begin{array}{c}\gamma_{\pm11}\left(u_{\pm}^{\delta},\omega_{\pm}^{\delta}\right)=D_1u_{\pm1}^{\delta},\ \gamma_{\pm22}\left(u_{\pm}^{\delta},\omega_{\pm}^{\delta}\right)=D_2u_{\pm2}^{\delta},\\ \gamma_{\pm12}\left(u_{\pm}^{\delta},\omega_{\pm}^{\delta}\right)=D_1u_{\pm2}^{\delta}-\omega_{\pm}^{\delta},\ \gamma_{\pm21}\left(u_{\pm}^{\delta},\omega_{\pm}^{\delta}\right)=D_2u_{\pm1}^{\delta}+\omega_{\pm}^{\delta},\end{array}$$

and $\sigma_{\pm ji}$ is the asymmetric stress tensor given by the linear law:

$$\begin{array}{c}\sigma_{\pm ji}\left(u_{\pm}^{\delta},\omega_{\pm}^{\delta}\right)=\left(\mu_{\pm}+\alpha_{\pm}\right)\gamma_{\pm ji}\left(u_{\pm}^{\delta},\omega_{\pm}^{\delta}\right)+\left(\mu_{\pm}-\alpha_{\pm}\right)\gamma_{\pm ij}\left(u_{\pm}^{\delta},\omega_{\pm}^{\delta}\right)\\ +\lambda_{\pm}\left(\sum_{k=1}^{2}\gamma_{\pm kk}\left(u_{\pm}^{\delta},\omega_{\pm}^{\delta}\right)\right)\delta_{ij};\ i,j=1,2\end{array}$$

where δ_{ij} is the Kronecker delta and $\mu_{\pm}$, $\lambda_{\pm}$, $\alpha_{\pm}$ are material constants satisfying the inequalities

$$\mu_{\pm}>0,\ 3\lambda_{\pm}+2\mu_{\pm}>0,\ \alpha_{\pm}>0.$$

It is well known (see [7]), that the transmission problem (P^{δ}) has a unique solution in the canonical Sobolev space

$$V=\left\{\begin{array}{c}\left(v_{\pm},\varphi_{\pm}\right)\in\mathbb{H}^1\left(\Omega_{\pm}\right)\times H^1\left(\Omega_{\pm}\right)=\left(H^1\left(\Omega_{\pm}\right)\right)^2\times H^1\left(\Omega_{\pm}\right);\\ v_-=0\text{ on }\Gamma_-,\ \varphi_-=0\text{ on }\Gamma_-,\\ v_{+1}=v_{-1}\text{ on }\Sigma,\ v_{+2}=v_{-2}\text{ on }\Sigma,\ \varphi_+=\varphi_-\text{ on }\Sigma\end{array}\right\}.$$

When the thickness δ of the thin layer is small enough the numerical solution of the transmission problem (P^δ) is usually very diffcult to calculate. In fact the small thickness of the thin layer Ω_+^δ generates numerical instabilities during the numerical computation of the solution. We then model the effect of the thin layer Ω_+^δ on the fixed domain Ω_- to bring back the transmission problem to an equivalent boundary limits problem set in the fixed domain Ω_- called impedance problem , i.e. we replace the system in Ω_+^δ, the transmission conditions on Σ and the boundary conditions on Γ_+^δ by an approximate boundary condition on Σ called approximate impedance condition. This condition will be established by a method based on the techniques of asymptotic expansion with scaling.

2. **The concept of impedance of a thin layer and the main results**

The parameter is δ supposed to be small enough, we will replace the transmission problem set in Ω^δ by a problem set just in the fixed domain Ω_-. For that we solve the following limits problem.

$$\left(P_+^\delta\right) : \begin{cases} \text{Equations (2) in } \Omega_+^\delta, \\ \text{Boundary conditions (3) on } \Gamma_+^\delta, \\ u_+^\delta = \psi^\delta \text{ on } \Sigma,\ \omega_+^\delta = \phi^\delta \text{ on } \Sigma, \end{cases}$$

if $\left(u_+^\delta, \omega_+^\delta\right)$ is the solution of this problem. We set

$$T_\delta\left(\left(\psi^\delta, \phi^\delta\right) = \left(u_+^\delta, \omega_+^\delta\right)_{|\Sigma}\right) := \left(\left[\sigma_+\left(u_+^\delta, \omega_+^\delta\right)\right]_{|\Sigma}^T n,\ \left[\left(\nu_+ + \epsilon_+\right)\frac{\partial \omega_+^\delta}{\partial n}\right]_{|\Sigma}\right),$$

using the transmission conditions on Σ (5), we obtain

$$T_\delta\left(\left(u_-^\delta, \omega_-^\delta\right)_{|\Sigma}\right) = \left(\left[\sigma_-\left(u_-^\delta, \omega_-^\delta\right)\right]_{|\Sigma}^T n,\ \left[\left(\nu_- + \epsilon_-\right)\frac{\partial \omega_-^\delta}{\partial n}\right]_{|\Sigma}\right),$$

and the impedance problem on Ω_- is written

$$\left(P_-^\delta\right) : \begin{cases} \text{Equations in } \Omega_-, \\ \text{Boundary conditions } \Gamma_-, \\ \left(\left[\sigma_-\left(u_-^\delta, \omega_-^\delta\right)\right]_{|\Sigma}^T n,\ \left[\left(\nu_- + \epsilon_-\right)\frac{\partial \omega_-^\delta}{\partial n}\right]_{|\Sigma}\right) = T_\delta\left(\left(u_-^\delta, \omega_-^\delta\right)_{|\Sigma}\right) \text{ on } \Sigma. \end{cases}$$

Since the exact impedance operator T_δ is not reachable for general geometric case, we will just prove that it can be approximated by $T_{*\delta}$ defined by:

$$T_{*\delta}\left(\psi^\delta, \phi^\delta\right) = \delta\left(C_1\left(\psi^\delta, \phi^\delta\right),\ C_2\left(\psi^\delta, \phi^\delta\right),\ C_3\left(\psi^\delta, \phi^\delta\right)\right)$$

with

$$\begin{aligned} C_1\left(\psi^\delta, \phi^\delta\right) &= \frac{4\mu\left(\lambda+\mu\right)}{\lambda+2\mu}\frac{\partial}{\partial s}\left(\frac{\partial}{\partial s}\psi_\tau^\delta + \psi_n^\delta R(s)\right) + \frac{4\alpha\mu}{\mu+\alpha}R(s)\left(\frac{\partial}{\partial s}\psi_n^\delta - R(s)\psi_\tau^\delta - \phi^\delta\right), \\ C_2\left(\psi^\delta, \phi^\delta\right) &= -R(s)\frac{4\mu\left(\lambda+\mu\right)}{\lambda+2\mu}\left(\frac{\partial}{\partial s}\psi_\tau^\delta + \psi_n^\delta R(s)\right) + \frac{4\alpha\mu}{\mu+\alpha}\frac{\partial}{\partial s}\left(\frac{\partial}{\partial s}\psi_n^\delta - R(s)\psi_\tau^\delta - \phi^\delta\right), \\ C_3\left(\psi^\delta, \phi^\delta\right) &= \left(\frac{\partial}{\partial s}\psi_n^\delta - R(s)\psi_\tau^\delta - \phi^\delta\right) + \left(\nu_+ + \epsilon_+\right)\frac{\partial^2\phi^\delta}{\partial s^2}, \end{aligned}$$

where $R\ (s)$ is the radius of curvature of Σ at the point $m \in \Sigma$ defined by the curvilinear abscissa s. If $M \in \Omega_+^\delta$, then

$$u\,(M) = u(m, z) = u(s, z) = u_s(s, z)\tau(s) + u_n(s, z)n(s)$$

with $\tau(s)$ is the tangent vector to Σ at the point m and $n(s)$ is the unit vector normal at m obtained by carrying out a rotation of $(-\pi/2)$ of the vector $\tau(s)$.

The solution $(u_-^\delta, \omega_-^\delta)$ of (P^δ) in Ω_- is then approximated by the solution $(u_{-*}^\delta, \omega_{-*}^\delta)$ of the following well posed approximate impedance problem $\left(P_{-*}^\delta\right)$ given by:

$$\left(P_{-*}^\delta\right): \begin{cases} \operatorname{div}\left[\sigma_-\left(u_{-*}^\delta, \omega_{-*}^\delta\right)\right]^T = -f_{-1} \text{ in } \Omega_- \\ \left(\nu_- + \epsilon_-\right)\Delta\omega_{-*}^\delta + \sigma_{-12}\left(u_{-*}^\delta, \omega_{-*}^\delta\right) - \sigma_{-21}\left(u_{-*}^\delta, \omega_{-*}^\delta\right) = g_- \\ u_{-*}^\delta = 0 \text{ on } \Gamma_- \quad , \quad \omega_{-*}^\delta = 0 \text{ on } \Gamma_- \\ \left(\left[\left[\sigma_=\left(u_{-*}^\delta, \omega_{-*}^\delta\right)\right]^T n\right]_{|\Sigma}, \left[\left(\nu_- + \epsilon_-\right)\frac{\partial\omega_-^\delta}{\partial n}\right]_{|\Sigma}\right) = T_{*\delta}\left(u_{-*|\Sigma}^\delta, \omega_{-*|\Sigma}^\delta\right) \text{ on } \Sigma. \end{cases}$$

And by using the techniques of asymptotic expansion, we prove the following result:

Theorem 2.1. *The problem* $\left(P_{-*}^\delta\right)$ *has a unique solution in the space*

$$V_* = \left\{ \begin{array}{c} \left(v_-, \varphi_-\right) \in \mathbb{H}^1\left(\Omega_-\right) \times H^1\left(\Omega_-\right) = \left(H^1\left(\Omega_-\right)\right)^2 \times H^1\left(\Omega_-\right); \\ \left(\frac{\partial}{\partial s}v_{-s}, \frac{\partial}{\partial s}v_{-n}\right) \in \mathbb{L}^2\left(\Sigma\right) = \left(L^2\left(\Sigma\right)\right)^2, \ \frac{\partial}{\partial s}\varphi_- \in L^2\left(\Sigma\right), \\ v_- = 0 \ on\ \Gamma_-, \ \varphi_- = 0 \ on\ \Gamma_- \end{array} \right\},$$

and the following error estimate holds:

$$\left\|u_-^\delta - u_{-*}^\delta\right\|_{\mathbb{H}^1(\Omega_-)} + \left\|\omega_-^\delta - \omega_{-*}^\delta\right\|_{H^1(\Omega_-)} \leq C\delta^2,$$

where $C > 0$ *is a constant depending only on* f_-, g_-, $R(s)$ *and the elasticity coefficients.*

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APPROXIMATION IN LINÉAIRE INTEGRO-DIFFERENTIAL EQUATION

KHALISSA ZERAIBI AND BACHIR GAGUI

Abstract. In the present work,we applied the projection method of Galerkin on certain class of the intergo-differential equations with Volterra, for the purprose to determine the approximate solution,and to make comparison with the exact solution.

AMS Subject Classification: 35XX, 45XX35SQ99, 65D05, 65Z99,34K28.

Keywords and phrases. Integro-differentiel equation,projection method, Galerkin Principle, Interpolation.

1. Define the problem

To solve a physical phenomenon we need a mathematical model generally in the form of an ordinary differential equation, with partial, integral and integro-differential derivatives, the latter equation different from the two. That a theory said that this type of equations admits one or more solutions our goal is to solve some type of these equations using a tool based on projection theory, this thechnic is called the Galerkin method

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1

ASYMPTOTIC STABILITY OF SOLUTIONS FOR NONLINEAR DIFFERENTIAL EQUATIONS

SAFIA MEFTAH

Abstract. We interested by study of the existence, uniqueness and asymptotic stability of periodic solutions of nonlinear oscillators.

2010 Mathematics Subject Classification. 34C29, 34C25, 37H11.

Keywords and phrases. Limit cycle, averaging theory, polynomial differential system.

1. Define the problem

In this work, we proved the existence, uniqueness and asymptotic stability of periodic solutions of Van Der Pol equation in their general form as

$$\ddot{x} + \epsilon \left(ax^2 + b\dot{x}^2 + cx + d \right) \dot{x} + x = 0. \quad x(0) = A \text{ and } \dot{x}(0) = 0,$$

with $a, b, c, d \in \mathbb{R}$, $A > 0$, $0 < \epsilon << 1$.

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AVERAGE OPTIMAL CONTROL WITH NUMERICAL ANALYSIS OF CORONAVIRUS

ASMA LADJEROUD AND MERIEM LOUAFI

ABSTRACT. In this paper, we controlled the propagation of the Corona epidemic in the society by applying the average optimal control on the propagation equations of the virus, so our control was represented by the optimal control in free time.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 49J20, 49J21, 49N30, 49K20, 93C20, 93C41.

KEYWORDS AND PHRASES. optimal control, average control, numerical analysis for Covid19.

1. DEFINE THE PROBLEM

Let us consider the following mathematical model of propagation of the covid19 epidemic, which is a simplified model from Bats-Hosts-Reservoir-People (BHRP) [2] to Reservoir-People (RP) model:

$$(1)\qquad \begin{cases} \dfrac{ds}{dt} = n - ms - b_p s(i+ka) - b_w sw \\ \dfrac{de}{dt} = b_p s(i+ka) + b_w sw - (1-\delta)we - \delta w'e - me \\ \dfrac{di}{dt} = (1-\delta)we - (\gamma+m)i \\ \dfrac{da}{dt} = \delta w'e - (\gamma'+m)a \\ \dfrac{dr}{dt} = \gamma i + \gamma' a - mr \\ \dfrac{dw}{dt} = \varepsilon(i + ca - w) \end{cases}$$

such that,
δ the proportion of asymptomatic infection rate of people,
k the multiple of the transmissibility of A_p to that of I_P,
c the relative shedding coefficient of A_p compared to I_P.
n the birth rate of people,
m the death rate of people,
$\frac{1}{w}$ the incubation period of people,
$\frac{1}{w'}$ the latent period of people,
$\frac{1}{\gamma}$ the infectious period of symptomatic infection of people,
$\frac{1}{\gamma'}$ the infectious period of asymptomatic infection of people.

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BLOW-UP PHENOMENA FOR A VISCOELASTIC WAVE EQUATION WITH BALAKRISHNAN-TAYLOR DAMPING AND LOGARITHMIC NONLINEARITY

BELHADJI BOCHRA

Abstract. Our aim in this article is to study a nonlinear viscoelastic equation with strong damping, Balakrishnan-Taylor damping and logarithmic nonlinearity of the form

$$u_{tt}(x,t)-M(t)\Delta u(x,t)+\int_0^t g(t-s)\Delta u(x,s)ds+\mu_1 u_t(x,t)-\mu_2\Delta u_t(x,t) = u(x,t)|u(x,t)|^{p-2}\ln|u(x,t)| \quad (1)$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where g is a nonincreasing positive function. we establish a finite time blow-up result for the solution with positive initial energy as well as nonpositive initial energy.

1. introduction

In this paper, we are concerned with the following nonlinear wave equation with Balakrishnan-Taylor damping,

$$\begin{cases} u_{tt}(x,t) - M(t)\Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds \\ +\mu_1 u_t(x,t) - \mu_2\Delta u_t(x,t) = u(x,t)|u(x,t)|^{p-2}\ln|u(x,t)|, x\in\Omega, t>0 \\ u=0, \quad u\in\partial\Omega, t>0 \\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), x\in\Omega \end{cases} \quad (1.1)$$

Where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$ and $M(t) = a + b\|\nabla u\|_2^2 + \sigma\int_\Omega \nabla u(t)\nabla u_t(t)dx$,$a$,$b$,$\sigma$ are positive constants. We prove the blow-up result under the following suitable assumptions.

(A1): $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a differentiable and decreasing function such that

$$g(t) \geq 0, \quad a - \int_0^\infty g(s)ds = l > 0 \quad (1.2)$$

(A2): There exists a constant $\xi > 0$ such that

$$g'(t) \leq -\xi g(t), \quad t \geq 0 \quad (1.3)$$

(A3): The exponent p satisfies

$$2 < p < \infty \text{ for } n = 1,2 \quad \text{and} \quad 2 < p < \frac{2(n-1)}{n-2} \text{ for } n \geq 3 \quad (1.4)$$

Let c_q be the best constants in the Poincaré type inequality

$$\|u\|_q \leq c_q\|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega) \quad (1.5)$$

for $2 \leq q \leq \infty$ if $n = 1,2$ or $2 \leq q \leq \frac{2n}{n-2}$ if $n \geq 3$.

Key words and phrases. Blow-up, stable and unstable set, global solutions, viscoelastic equation,strong damping, Balakrishnan-Taylor damping.

Lemma 1.1. *For all $q > 0$,*

$$|s^q \ln s| \le \frac{1}{eq}, \text{ for } 0 < s < 1 \quad \text{and} \quad 0 \le s^{-q} \ln s \le \frac{1}{eq} \text{ for } s \ge 1 \tag{1.6}$$

Lemma 1.2. [1] *Let $L(t)$ be a positive, twice differentiable function satisfying the inequality*

$$L(t)L''(t) - (1+\delta)(L'(t))^2 \ge 0 \tag{1.7}$$

with some $\delta > 0$. If $L(0) > 0$ and $L'(0) > 0$ then there exists $T^* \le L(0)/\delta L'(0)$ such that $\lim_{t\to T^{-*}} L(t) = \infty$.

We define the energy associated with the solution of system (1.1) by

$$\begin{aligned} E(t) := &\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(a - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{b}{2}\|\nabla u(t)\|_2^4 + \frac{1}{2}\left(g \circ \nabla u\right)(t) \\ &- \frac{1}{p}\int_\Omega |u(x,t)|^p \ln|u(x,t)|dx + \frac{1}{p^2}\|u(t)\|_p^p \end{aligned} \tag{1.8}$$

The energy functional defined by (1.8) is a non-increasing function on $[0,T]$ and

$$\frac{d}{dt}E(t) + \mu_1\|u_t(t)\|_2^2 + \mu_2\|\nabla u_t(t)\|_2^2 = -\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 + \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \tag{1.9}$$

and hence

$$E(t) + \mu_1\int_0^t \|u_t(s)\|_2^2 ds + \mu_2\int_0^t \|\nabla u_t(s)\|_2^2 ds \le E(0), \quad 0 \le t \le T \tag{1.10}$$

In order establish the blow-up of the weak solution for problem (1.1), we set the following energy and Nehari's functionals:

$$J(u) = \frac{1}{2}\left(a - \int_0^\infty g(s)ds\right)\|\nabla u(t)\|_2^2 - \frac{1}{p}\int_\Omega |u(x,t)|^p \ln|u(x,t)|dx \tag{1.11}$$

$$I(u) = \left(a - \int_0^\infty g(s)ds\right)\|\nabla u(t)\|_2^2 - \int_\Omega |u(x,t)|^p \ln|u(x,t)|dx \tag{1.12}$$

Therfore

$$J(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(a - \int_0^\infty g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{p}I(u) \tag{1.13}$$

Define the Nehari's manifold

$$\mathcal{N} = \{u \in H_0^1(\Omega) | I(u) = 0, \|\nabla u\|_2 \ne 0\} \tag{1.14}$$

Next, let us define the stable set W and the unstable set V as follows

$$W = \{u \in H_0^1(\Omega) | I(u) > 0, J(u) < d\} \cup \{0\}, \tag{1.15}$$

$$V = \{u \in H_0^1(\Omega) | I(u) < 0, J(u) < d\}, \tag{1.16}$$

Where d is the depth of the potential well that can be characterized by

$$d = \inf_{0\ne u\in H_0^1(\Omega)} \sup_{\lambda\ge 0} J(\lambda u) = \inf_{u\in\mathcal{N}} J(u) \tag{1.17}$$

Lemma 1.3. *The depth d of the potential well W is positive.*

Lemma 1.4. *For any $u \in H_0^1(\Omega)$, $\|\nabla u\|_2 \ne 0$ there exists a unique $\lambda^\star = \lambda^\star(u) > 0$ such that*

(i): $\lim_{\lambda\longrightarrow o^+} J(\lambda u) = 0$, $\lim_{\lambda\longrightarrow +\infty} J(\lambda u) = -\infty$

(ii): *$J(\lambda u)$ is increasing on $0 < \lambda \le \lambda^*$, decreasing on $\lambda^* \le \lambda < +\infty$ and takes its maximum at $\lambda = \lambda^\star$ where* $\frac{d}{d\lambda}J(\lambda u)|_{\lambda=\lambda^*} = 0$

(iii): *$I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < +\infty$ and $I(\lambda^* u) = 0$*

Proof. For $\lambda > 0$, we have

$$\begin{aligned}\frac{\partial}{\partial\lambda}J(\lambda u) &= \lambda\left(\left(a - \int_0^\infty g(s)ds\right)\|\nabla u(t)\|_2^2 - \lambda^{p-2}\int_\Omega |u(x,t)|^p \ln|u(x,t)|dx - \lambda^{p-2}\ln\lambda\int_\Omega |u(x,t)|^{p+1}dx\right)\\ &:= \lambda K(\lambda u)\end{aligned} \tag{1.18}$$

The function $K(\lambda u)$ is increasing on $0 < \lambda < \lambda_1$ and decreasing for $\lambda > \lambda_1$ where

$$\lambda_1 = exp\left(\frac{(p-2)\int_\Omega |u(x,t)|^p \ln|u(x,t)|dx + \int_\Omega |u(x,t)|^p dx}{(2-p)\int_\Omega |u(x,t)|^p dx}\right) < 1 \tag{1.19}$$

We remark from the definition of the function $K(\lambda u)$ that

$$\lim_{\lambda\longrightarrow o^+} K(\lambda u) = \left(a - \int_0^\infty g(s)ds\right)\|\nabla u(t)\|_2^2 > 0, \lim_{\lambda\longrightarrow +\infty} K(\lambda u) = -\infty \tag{1.20}$$

Therfore, there exists a unique $\lambda^* > \lambda_1$ such that $K(\lambda^\star u) = 0$ and we have (ii). Since $I(\lambda u) = \lambda\frac{\partial J(\lambda u)}{\partial\lambda}$ which is verified by a direct computation then one has (iii). □

Lemma 1.5. [2] *Let (**A**1) and (**A**2) hold. If $I(u_0) < 0$ and $E(0) < d$, then the solution of the problem* (1.1) *satisfies*

$$I(u(t)) < 0 \text{ and } E(t) < d, \quad 0 \le t \le T \tag{1.21}$$

Theorem 1.6. *Let (**A**1) and (**A**2) hold. Suppose that $I(u_0) < 0$ and $E(0) = \alpha d$, where $\alpha < 1$, and the kernel function g satisfies*

$$\int_0^\infty g(s)ds \le \frac{p-2}{a(p-2) + 1/((1-\tilde\alpha)^2 p + 2\alpha(1-\tilde\alpha))} \tag{1.22}$$

where $\tilde\alpha = \max(0,\alpha)$. Moreover, suppose that $\int_\Omega u_0(x)u_1(x)dx > 0$ when $E(0) = 0$. Then the solution u of problem (1.1) *blows up in finite time.*

Proof. Let us define the functional L as follows

$$\begin{aligned}L(t) &= \|u(t)\|_2^2 + \frac{\sigma}{2}\left(\int_0^t \|\nabla u(t)\|_2^4 ds + (T-t)\|\nabla u_0\|_2^4\right) + \mu_1\int_0^t \|u(s)\|_2^2 ds\\ &+ \mu_2\int_0^t \|\nabla u(s)\|_2^2 ds + c(t+T_0)^2\end{aligned} \tag{1.23}$$

where $T_0 > 0$ and $c \ge 0$, which are specified later. Hence

$$L(t) > 0 \text{ on } 0 \le t \le T, \tag{1.24}$$

and

$$L'(t) = 2(u(t), u_t(t)) + \frac{\sigma}{2}\int_0^t \frac{d}{ds}\|\nabla u(s)\|_2^4 ds + 2\mu_1 \int_0^t \int_\Omega u_t(t)u(t)dxds \\ + 2\mu_2 \int_0^t \int_\Omega \nabla u_t(t)\nabla u(t)dxds + 2c(t+T_0) \tag{1.25}$$

and from the first equation in (1.1) we have

$$\begin{aligned} L''(t) &= 2\|u_t(t)\|_2^2 + 2\int_\Omega u(t)u_{tt}(t) + \sigma\left(\int_\Omega \nabla u_t(t)\nabla u(t)\right)\|\nabla u(t)\|_2^2 \\ &= 2\|u_t(t)\|_2^2 + 2\int_\Omega |u(x,t)|^p \ln|u(x,t)|dx - 2a\|\nabla u(t)\|_2^2 - 2b\|\nabla u(t)\|_2^4 \\ &+ 2\int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla u(t)dxds + 2c \end{aligned} \tag{1.26}$$

Then

$$L''(t) = 2\|u_t(t)\|_2^2 + 2\int_\Omega |u(x,t)|^p \ln|u(x,t)|dx - 2\left(a - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 \\ - 2\int_0^t g(t-s)(\nabla u(t), \nabla u(t) - \nabla u(s))ds - 2b\|\nabla u(t)\|_2^4 + 2c \tag{1.27}$$

Therefore, using the definition of $L(t)$, we get

$$\begin{aligned} L(t)L''(t) - \frac{p+2}{4}(L'(t))^2 = L(t)L''(t) + (p+2)\Big(&F(t) - (L(t) - (T-t)(\mu_1\|u_0(t)\|_2^2 + \mu_2(\nabla\|u_0(t)\|_2^2) \\ &+ \frac{\sigma}{2}\|u_0(t)\|_2^4))\Big(\|u_t(t)\|_2^2 + \frac{\sigma}{2}\int_0^t \|\nabla u(s)\|_2^4 ds + \mu_1\int_0^t \|u_t(s)\|_2^2 ds \\ &+ \mu_2\int_0^t \|\nabla u_t(s)\|_2^2 ds\Big) + c\Big) \end{aligned} \tag{1.28}$$

where the function $F(t)$ is defined by

$$\begin{aligned} F(t) = &\Big[\|u(t)\|_2^2 + \frac{\sigma}{2}\int_0^t \|\nabla u(s)\|_2^4 ds + \mu_1\int_0^t \|u(s)\|_2^2 ds \\ &+ \mu_2\int_0^t \|\nabla u(s)\|_2^2 ds + c(t+T_0)^2\Big]\Big[\|u(t)\|_2^2 + \frac{\sigma}{2}\int_0^t \|\nabla u_t(s)\|_2^4 ds + \mu_1\int_0^t \|u_t(s)\|_2^2 ds \\ &+ \mu_2\int_0^t \|\nabla u_t(s)\|_2^2 ds + c\Big] - \Big[\int_\Omega u_t(t)u(t)dx + \frac{\sigma}{4}\int_0^t \frac{d}{ds}\|\nabla u(s)\|_2^4 ds + \mu_1\int_0^t\int_\Omega u_t(s)u(s)dxds \\ &+ \mu_2\int_0^t\int_\Omega \nabla u_t(s)\nabla u(s)dxds + c(t+T_0)\Big]^2 \end{aligned} \tag{1.29}$$

By Cauchy-Schwarz inequality, we read the following differential inequality

$$L(t)L''(t) - \frac{p+2}{4}(L'(t))^2 \geq L(t)K(t), \quad \forall t \in [0,T] \tag{1.30}$$

Where

$$K(t) = -p\|u_t\|_2^2 - 2\left(a - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 - \frac{b}{2}\|\nabla u(t)\|_2^4 + 2\int_\Omega |u(x,t)|^p \ln|u(x,t)|dx$$
$$- (p+2)\Big[\mu_1\int_0^t \|u_t(s)\|_2^2 ds + \mu_2\int_0^t \|\nabla u_t(s)\|_2^2 ds + \frac{\sigma}{2}\int_0^t \|\nabla u_t(s)\|_2^4 ds\Big]$$
$$- 2\int_\Omega \nabla u(t)\left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))ds\right)dx - pc \tag{1.31}$$

From (1.10) and (1.8) we can write

$$K(t) \geq -2pE(0) + (p-2)\left(a - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + p(g\circ\nabla u)(t) + \frac{p-2}{2}b\|\nabla u(t)\|_2^4$$
$$+ (p-2)\left(\mu_1\int_0^t \|u_t(s)\|_2^2 ds + \mu_2\int_0^t \|\nabla u_t(s)\|_2^2 ds + \frac{\sigma}{4}\int_0^t \|\nabla u_t(s)\|_2^4\right)$$
$$- 2\int_\Omega \nabla u(t)\left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))ds\right)dx - pc$$
$$\geq -2pE(0) + \left((p-2)a - (p-2+\frac{1}{\varepsilon})\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + (p-\varepsilon)(g\circ\nabla u)(t)$$
$$+ \frac{2}{p}\|u(t)\|_p^p + \mu_2(p-2)\int_0^t \|\nabla u_t(s)\|_2^2 ds - pc \tag{1.32}$$

where $\varepsilon > 0$. Now we consider the initial energy $E(0)$ divided into three cases,

case 1: $E(0) < 0$

Taking $\varepsilon = p$ in (1.32) and choosing $0 < c \leq -2E(0)$ we have

$$K(t) \geq p(-2E(0) - b) + \left((p-2)a - (p-2+\frac{1}{p})\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2$$
$$+ \frac{2}{p}\|u(t)\|_p^p + \mu_2(p-2)\int_0^t \|\nabla u_t(s)\|_2^2 ds \geq 0 \tag{1.33}$$

Choosing T_0 large enough we get $L'(0) = 2\int_\Omega u_0(x)u_1(x)dx + 2cT_0 > 0$.

case 2: $E(0) = 0$

Taking $\varepsilon = p$ in (1.32) and choosing $c = 0$ we have

$$K(t) \geq \left((p-2)a - (p-2+\frac{1}{p})\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2$$
$$+ \frac{2}{p}\|u(t)\|_p^p + \mu_2(p-2)\int_0^t \|\nabla u_t(s)\|_2^2 ds \geq 0 \tag{1.34}$$

case 2: $0 < E(0) < d$

Taking $\varepsilon = (1-\alpha)p + 2\alpha$ in (1.32), we find

$$K(t) \geq -2pE(0) + \left((p-2) - \left(p - 2 + \frac{1}{(1-\alpha)p + 2\alpha}\right)\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2$$
$$+ \alpha(p-2)(g\circ\nabla u)(t) + \frac{2}{p}\|u(t)\|_p^p + \mu_2(p-2)\int_0^t \|\nabla u_t(s)\|_2^2 ds - pc \tag{1.35}$$

and from (1.22), it follows that

$$K(t) \geq -2pE(0) + \alpha(p-2)\left(a - \int_0^\infty g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{2}{p}\alpha\|u(t)\|_p^p - pc \quad (1.36)$$

From Lemma(1.4) and using (1.11)-(1.12) we deduce that

$$d \leq J(\lambda^* u(t)) < \frac{p-2}{2p}\left(a - \int_0^\infty g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{p^2}\|u(t)\|_p^p \quad (1.37)$$

Since u is continious on $[0,T]$ then there exists $d_1 > 0$ such that

$$d + d_1 < \frac{p-2}{2p}\left(a - \int_0^\infty g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{p^2}\|u(t)\|_p^p \quad (1.38)$$

From this and (1.36), we get

$$\begin{aligned} K(t) &\geq -2p\alpha d + 2\alpha p\left(\frac{p-2}{2p}\left(a - \int_0^\infty g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{p^2}\|u(t)\|_p^p\right) - pc \\ &> 2\alpha p d_1 - pc \end{aligned} \quad (1.39)$$

Hence for c small enought we get $L(t) \geq 0$.Choosing T_0 large enough we get

$$L'(0) = 2\int_\Omega u_0(x)u_1(x)dx + 2cT_0 > 0$$

Thus, we conclude from Lemma(1.2) that $\lim_{t\to T^{-*}} L(t) = \infty$ which implies that $\lim_{t\to T^{-*}} \|\nabla u(t)\|_2^2 = \infty$ □

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Blow-up results for fractional damped wave models with non-linear memory

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Keywords: wave equations, nonlinear memory, test functions, weak solutions

ABSTRACT

This paper is devoted to find the critical exponent in Fujita sense and to prove the blow-up results of solution of the following Cauchy problem

$$u_{tt} - \Delta u + D^{\sigma}_{0|t} u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau, \cdot)|^p \, d\tau, \tag{1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n. \tag{2}$$

where $p > 1,\ 0 < \gamma < 1$ and Δ is the usual Laplace operator, $\sigma \in]0,1[$ and $D^{\sigma}_{0|t}$ is the right hand side fractional operator of Riemann-Liouville, by using the test function method.

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CONVERGENCE OF FINITE VOLUME MONOTONE SCHEMES FOR STOCHASTIC GENERALIZED BURGERS EQUATION ON A BOUNDED DOMAINS

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ABSTRACT. This paper is devoted to the study of finite volume methods for the discretization of the generalized Burgers equation with additive stochastic force defined on a bounded domain D of $\mathbb{R}$ with Dirichlet boundary conditions and a given initial data in $L^{\infty}(D)$. We introduce a notion of stochastic measure-valued entropy solution which generalizes the concept of weak entropy solution introduced by F.Otto for such kind of hyperbolic bounded value problems in the deterministic case. We prove that the numerical solution converges to the stochastic measure-valued entropy solution of the continuous problem under a stability condition on the time and space steps, this also proof the existence of a measure-valued entropy weak solution.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 60H15, 35L60, 60H40, 65M08.

KEYWORDS AND PHRASES. Stochastic Burgers equation, first-order hyperbolic equation, space-time white noise, finite volume method, monotone scheme, Dirichlet boundary conditions.

1. DEFINE THE PROBLEM

We wish to find an approximate solution to the following nonlinear scalar conservation law with a stochastic additive force, posed over a bounded domain D with initial condition and Dirichlet boundary conditions:

$$(1)\qquad \begin{cases} \frac{\partial u(\omega,t,x)}{\partial t}+\frac{\partial (f(u(\omega,t,x)))}{\partial x} & =\frac{\partial W}{\partial t} \quad \text{in } \Omega\times]0,T[\times D,\\ u(\omega,0,x) & =u_0(x) \quad \omega\in\Omega,\ x\in D,\\ u(\omega,t,x) & =0 \quad \omega\in\Omega,\ t\in]0,T[,\ x\in\partial D \end{cases}$$

where $D=]0,2\pi[$, $T>0$ and W is a cylindrical Wiener process.
Recall that the cylindrical Wiener process can be written as

$$(2)\qquad W(t,x)=\sum_{k=1}^{\infty}\beta_k(t)\,e_k(x)$$

where $\{e_k\}$ is any orthonormal basis of $L^2(0,2\pi)$ and $\{\beta_k\}$ is a sequence of mutually independent real Brownian motions in a fixed probability space $(\Omega,\mathcal{F},\mathbb{P})$ adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

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CALCULATING THE H_∞ NORM FOR A NEW CLASS OF FRACTIONAL STATE SPACE SYSTEMS

AMINA FARAOUN AND DJILLALI BOUAGADA

Abstract. This paper outlines a pratical algorithm to calculate the H_∞ norm of an othor class of fractional linear state space systems as an extension of the work in [1]. This method was obtained from using a parahermitian matrix function and level sets of maximum singular value of the transfer function. Some examples are given to illustrate the approach.

2010 Mathematics Subject Classification. 15A30, 37N35, 26A33, 44A10.

Keywords and phrases. Fractional systems, Singular systems, H_∞ norm, Level sets.

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CHAOTIC BEHAVIOR IN THE PRODUCT OF GENERATING FUNCTIONS

NOURA LOUZZANI AND **ABDELKRIM BOUKABOU**

Abstract. In this paper we propose a generating function of binary product of Fibonacci numbers with Mersenne Lucas numbers that can show a typical period-doubling cascade to chaos. In this context, the bifurcation diagram and Lyapunov exponent proved that the proposed generating function is a deterministic system that exhibits chaotic behavior for certain ranges of the control parameters.
2010 Mathematics Subject Classification. 33C65, 34A26, 94A60.
Keywords and phrases. Generating function, Mersenne Lucas numbers, Chaos, Lyapunov Exponent, Bifurcation Diagram.

1. Define the problem

We propose in this paper a generating function of the binary product of Fibonacci numbers with Mersenne Lucas numbers that can show a typical period-doubling cascade to chaos. To derive the related generating function of the binary product of (p, q) Fibonacci numbers with Mersenne Lucas numbers, we introduce intuitive modifications as proposed in [3, 4] , by considering a new operator in order to derive some new symmetric properties of the Fibonacci numbers and Mersenne Lucas numbers . Moreover, the chaotic behavior is observed by analyzing the bifurcation diagram and quantifying the Lyapunov exponents for different parameter values. As an application, this proposed generating function is used as a chaos-related fields of interest.

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QUALITATIVE ANALYSIS OF AN EPIDEMIC MODEL WITH NONLINEAR INCIDENCE RATE IN THE TIME OF COVID-19

NADIA MOHDEB

Abstract. In this paper we propose and study an epidemic model with nonlinear incidence rate, describing some factors effect (protection, exposure, immigration, social distancing, vaccination) on the spread of certain diseases on the community like the novel coronavirus COVID-19. The dynamical behavior of the proposed model is examined. We investigate the existence and stability of the disease-free equilibrium and the endemic equilibrium. The existence of a limit cycle is studied. Simulations of the model are performed to illustrate and support the theoretical results.

2010 Mathematics Subject Classification. 34A34, 34C23, 34C25, 92C60, 92D25.

Keywords and phrases. Epidemic model, Covid-19, Nonlinear incidence, Stability, Psychological effect, Cure rate.

1. Define the problem

On March 11, 2020, the coronavirus COVID-19 has emerged in the world and World Health Organization has declared it a pandemic. Today, the spread of this disease is impressive and has widespread socio-economic-political impacts. Using mathematical models for the description of infectious disease provides information about the transmission of a disease in the community. The first mathematical model of infectious disease was formulated in 1927 by Karmack and Mckendrick [5].

To have a clear understanding of the COVID-19 transmission dynamics and in order to reduce the spread, many researchers model the disease to figure out the properties [2], [4], [6], [7], [9]-[11]. Recently in [8], the author have constructed the following mathematical model, inspired from the classic Lotka-Voltera model [1],

$$(1)\qquad \begin{cases} \dfrac{dh(t)}{dt} = ah(t) - bh(t)i(t) + ei(t) \\ \dfrac{di(t)}{dt} = bh(t)i(t) + ci(t) - di(t) - ei(t) \end{cases}$$

The model (1) is composed of two compartments, healthy and infected; at time t, the healthy individual population is given by $h(t)$ and the infected by $i(t)$. The parameters a, b, c, d, and e are positive constants: b is the infection rate ($b = 1-$ protection rate), d is the death rate, and e is the cure rate. a is the immigration rate of healthy individuals, and c is that of infected individuals; immigration has a severe impact of the spreading of this virus.

Capasso and Serio [3] used, to model cholera epidemic spread in Bari in 1973, a nonlinear saturated incidence rate. This incidence rate seems more reasonable than the bilinear incidence rate, because it includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate by choosing suitable parameters. In order to represent the nonlinear incidence rate of the COVID-19 outbreak, we consider in this work, the model (1) and we use the function $kI/(1+\alpha I)$. The parameters k and α are positive constants, where kI measures the infection force of the disease and $1/(1+\alpha I)$ describes the inhibition effect, interpreted as psychological effect, usually forced by governmental measures.

In this work, basic results are given; we study the existence of equilibria of the model, their types and their stability. Bifurcations and existence of periodic solutions for the model are examined. We simulate some results using different values of the parameters and plotted the outcomes. A conclusion about some factors effect on the spread of the novel coronavirus COVID-19 on the community, is presented.

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SLIP DEPENDENT FRICTION IN QUASISTATIC VISCOPLASTICITY.

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Abstract. We consider a mathematical model which describes the quasistatic contact between a viscoplastic body and an obstacle the so-called foundation. The contact is modelled with a version of Coulomb's law with slip-dependent friction in which the normal stress is prescribed on the contact surface. Under appropriate assumptions, we provide a variational formulation to the mechanical problem for which we prove the existence of a weak solution.The proof is based on the time-discretization method, the Banach fixed point theorem and arguments of monotonicity, compactness and lower semicontinuity.

2010 Mathematics Subject Classification. 74C10, 74M15, 49J40, 74H20, 74A55.

Keywords and phrases. Viscoplastic material, Coulomb's law of dry friction, slip dependent coefficient of friction, quasistatic, time-discretization method, variational inequalities.

The aim of this paper is to provide a variational analysis in the study of the frictional contact between a viscoplastic body and a foundation. The physical setting is as follows. A deformable body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ (with d=2, 3). The material's behaviour is modelled with a rate-type constitutive law and the process is quasistatic in the time interval of interest $[0, T]$. We assume that the boundary Γ of the domain Ω is Lipschitz continuous, and it is partitioned into three disjoint measurable parts Γ_1, Γ_2, Γ_3, such that meas$(\Gamma_1) > 0$. The body is clamped on Γ_1 and therefore the displacement field vanishes there, while volume forces of density f_0 act in Ω and surface tractions of density f_2 act on Γ_2. The body is supposed to be in frictional contact over Γ_3 with a foundation. The contact is described by a version of Coulomb's law of dry friction with slip dependent friction in which the normal stress is prescribed on the contact surface. Under the above assumptions, the classical formulation of our problem is the following.

Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$ and a stress field $\sigma : \Omega \times [0,T] \to \mathbb{S}^d$ such that

$$\dot{\sigma} = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{B}(\sigma, \varepsilon(u)),\ in\ \Omega \times (0,T), \tag{1}$$

$$Div\sigma + f_0 = 0,\ in\ \Omega \times (0,T), \tag{2}$$

$$u = 0,\ on\ \Gamma_1 \times (0,T), \tag{3}$$

$$\sigma\nu = f_2,\ on\ \Gamma_2 \times (0,T), \tag{4}$$

$$-\sigma_\nu = S,\ on\ \Gamma_3 \times (0,T), \tag{5}$$

$$\begin{cases} |\sigma_\tau| \leq S\mu\left(|u_\tau|\right), \\ |\sigma_\tau| < S\mu\left(|u_\tau|\right) \Rightarrow \dot{u}_\tau = 0, \\ |\sigma_\tau| = S\mu\left(|u_\tau|\right) \Rightarrow \exists \lambda \geq 0, \\ \text{such that } \ \sigma_\tau = -\lambda \dot{u}_\tau, \end{cases} on\ \Gamma_3 \times (0,T), \tag{6}$$

$$u(0) = u_0,\ \sigma(0) = \sigma_0\ in\ \Omega. \tag{7}$$

Equation (1) represents the rate-type viscoplastic constitutive law. Equation (2) represents the equilibrium equation posed on the domain Ω. Conditions (3)-(4) are the displacement-traction boundary conditions where $\sigma\nu$ represents the Cauchy stress vector. Conditions (5)-(6) characterize the contact boundary conditions on Γ_3, where (5) indicates that the normal stress is prescribed on the contact surface. Condition (6) states that the tangential shear σ_τ is bounded by the normal stress S multiplied by the value of the friction coefficient $\mu\left(|u_\tau|\right)$, such that sliding takes place only when the equality holds and the friction stress in this case is proportional and opposed to the tangential velocity. Finally (7) are the initial conditions.

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EXISTENCE OF SOLUTIONS FOR NONLINEAR HILFER-KATUGAMPOLA FRACTIONAL DIFFERENTIAL INCLUSIONS

MOHAMMED SAID SOUID

ABSTRACT. This paper is concerned with the existence of solutions for nonlinear initial value problem for fractional differential inclusions in weighted space involving the Hilfer-Katugampola fractional derivative. Both cases of convex and nonconvex valued right hand sides are considered.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 26A33, 34A08.

KEYWORDS AND PHRASES. Hilfer-Katugampola fractional derivative, set-valued maps, differential inclusions, fixed point.
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1. Introduction

In this work we deal with the existence of solutions for the initial value problem (IVP for short), for Hilfer-Katugampola fractional differential inclusions

$$ {}^{\rho}\mathcal{D}_{a^+}^{\nu_1,\nu_2} z(t) \in \mathcal{M}(t, z(t), {}^{\rho}\mathcal{I}_{a^+}^{\alpha} z(t)), \quad t \in \mathcal{J} := [a, b], \tag{1} $$

$$ ({}^{\rho}\mathcal{I}_{a^+}^{1-\nu} z)(a) = \lambda, \quad \lambda \in \mathbb{R}, \ \nu = \nu_1 + \nu_2(1 - \nu_1), \tag{2} $$

where $\nu_1 \in (0, 1)$, $\nu_2 \in [0, 1]$, $\rho > 0$, $\mathcal{M} : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is multivalued map, $(\mathcal{P}(\mathbb{R}))$ is the family of all nonempty subsets of $\mathbb{R}$, ${}^{\rho}\mathcal{D}_{a^+}^{\nu_1,\nu_2}$ is the Hilfer Katugampola fractional derivative of order ν_1 and type ν_2 and ${}^{\rho}\mathcal{I}_{a^+}^{\nu_1}$, ${}^{\rho}\mathcal{I}_{a^+}^{1-\nu}$ are Katugampola fractional integral of order ν_1 and $1 - \nu$, respectively with $a > 0$.

The rest of paper is organized as follows: In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following Sections. In Section 3, we present two results for existence solutions of the problem $(1) - (2)$, our first result is based of Bohnnenblust-Karlin fixed point theorem when the right hand side is convex valued, the second result is based on contraction multivalued maps given by Covitz and Nadler when the right hand side is nonconvex valued. An example is given in Section 4 to illustrate the application of our main results. These results can be considered as a contribution to this emerging field.

2. Main results

For the existence of solutions for our problem, we need the following auxiliary lemma.

Lemma 2.1. *(See* [9]*) Let* $\nu = \nu_1+\nu_2(1-\nu_1)$, *where* $0 < \nu_1 < 1,\ 0 \le \nu_2 \le 1$ *and* $\rho > 0$. *let* $g \in C_{1-\nu,\rho}(\mathcal{J})$. *A function* z *is a solution of the fractional integral equation*

$$z(t) = \frac{\lambda}{\Gamma(\nu)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\nu-1} + \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\nu_1-1}\frac{g(s)}{\Gamma(\nu_1)}ds.$$

if and only if z *is a solution of the fractional initial value problem*

$$^{\rho}\mathcal{D}_{a^+}^{\nu_1,\nu_2}z(t) = g(t), \quad t \in \mathcal{J},$$

$$(^{\rho}\mathcal{I}^{1-\nu}z)(a) = \lambda, \quad \lambda \in \mathbb{R},\ \nu = \nu_1+\nu_2(1-\nu_1).$$

2.1. **The convex case.** Now we are concerned with the existence of solutions for the problem (1) − (2) when the right hand side has convex values. For this, we assume that $\mathcal{M}$ is a compact and convex valued multivalued map.
Let us introduce the following assumptions:

(H1): $\mathcal{M} : \mathcal{J}\times\mathbb{R}\times\mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$; $(t,y,z) \longmapsto \mathcal{M}(t,y,z)$
(i): is measurable, with respect to t, for each $y,z \in \mathbb{R}$,
(ii): upper semicontinuous with respect to $(y,z) \in \mathbb{R}\times\mathbb{R}$, for $a.e.\ t \in \mathcal{J}$.
(H2): There exists a continuous function $\varphi :\ \mathcal{J} \to \mathbb{R}^+$ such that:

$$\|\mathcal{M}(t,y,z)\|_{\mathcal{P}} = sup\{|g| : g \in \mathcal{M}(t,y,z)\} \le \frac{\varphi(t)}{1+|y|+|z|},\ t \in \mathcal{J}\ and\ y,z \in \mathbb{R}.$$

(H3): There exist $p,\ q \in \mathcal{L}^\infty(\mathcal{J})$ such that

$$\mathcal{H}_d(\mathcal{M}(t,y,z),\mathcal{M}(t,\overline{y},\overline{z}) \le p(t)|y-\overline{y}|+q(t)|z-\overline{z}|\ \ for\ a.e.\ t \in \mathcal{J}\ and\ y,\ \overline{y},\ z,\ \overline{z}\ \in \mathbb{R}.$$

The first result is based on Bohnnenblust-Karlin fixed point theorem.

Theorem 2.2. *Assume that the assumptions* (***H1***) − (***H3***) *are satisfied. then the IVP* (1) − (2) *has at least one solution on* $\mathcal{J}$.

2.2. **The nonconvex case.** This subsection is devoted to proving the existence of solutions for (1) − (2) with a nonconvex valued right hand side. Our second result is based on contraction multivalued maps given by Covitz and Nadler.
Let us introduce the following assumption:

(H4): $\mathcal{M} : \mathcal{J}\times\mathbb{R}\times\mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ has the property that $\mathcal{M}(.,y,z) : \mathcal{J} \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable, and integrable bounded for each $y,z \in \mathbb{R}$.

Theorem 2.3. *Assume that the assumptions* (***H3***) − (***H4***) *are satisfied. If*

$$\Big[\frac{p^*}{\Gamma(\nu_1+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\nu_1} + \frac{q^*}{\Gamma(2\nu_1+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{2\nu_1}\Big] < 1. \tag{3}$$

Then the IVP (1) − (2) *has at least one solution* $z \in C_{1-\nu,\rho}(\mathcal{J})$.

3. Example

As an application of our results we consider the following fractional initial value problem,

$$^{\rho}\mathcal{D}_{a^+}^{\frac{1}{2},\frac{1}{2}}z(t) \in \mathcal{M}(t,z(t),{}^{\rho}\mathcal{I}_{a^+}^{\nu_1}z(t)), \quad t \in \mathcal{J} := [\frac{\pi}{2},\pi], \tag{4}$$

$$(^{\rho}\mathcal{I}^{\frac{1}{4}} z)(\frac{\pi}{2}) = (1 - \frac{\pi}{2}), \tag{5}$$

where $\rho > 0,\ \nu = \frac{3}{4}$.
Since all conditions of Theorem 2.2 are satisfied, the IVP (4) − (5) has at least one solution.

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ETUDE COMPARATIVE ENTRE DEUX MÉTHODES HYBRIDES DU GRADIENT CONJUGUÉ AVEC RECHERCHE LINÉAIRE INEXACTE

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Abstract. La méthode du gradient conjugué est l'une des méthodes les plus efficaces pour résoudre les systèmes linéaires de grande dimension ainsi que les problèmes d'optimisation non linéaire sans contraintes.

Dans ce travail, on a fait une étude comparative entre deux méthodes hybrides du gradient conjugué, parmi les plus récentes, la méthode **MMDL** et **MLSCD,** en utilisant différentes règles de rcherche linéaire inexacte. Les tests numériques ont été effectués sur plusieurs fonctions tests et pour différentes dimensions (n).
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Keywords and phrases. Optimisation sans contraintes, Méthode du gradient conjugué, Recherche linéaire inexacte, Convergence globale, Méthode hybride.

1. Position du problème

Soit le problème d'optimisation sans contraintes:

$$(P\ )\begin{cases} \min f(x) \\ x \in \mathbb{R}^n \end{cases}, \quad \text{où } f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

Parmi les méthodes les plus utilisées pour résoudre ce type de problèmes, on a la méthode du gradient conjugué (G.C). Cette méthode a été proposée en 1952 par **Hestenes** et **Steifel (HS)**, pour résoudre des systèmes linéaires avec des matrices définies positives ce qui est équivalent à la minimisation de fonctions quadratiques strictement convexes. Depuis, plusieurs mathématiciens ont étendu cette méthode pour résoudre des problèmes non linéaires de type (P), où f n'est pas convexe. Ceci a été réalisé pour la première fois en 1964 par **Fletcher** et **Reeves (FR)**, puis en 1969 par **Polak**, **Ribière** et **Polyak (PRP)**, et depuis plusieurs variantes de la méthode du gradient conjugué ont été proposées jusqu'à nos jours, telles que celle de CD, DY, DHSDL, DLSDL.

Toutes ces méthodes génèrent une suite $\{x_k\}_{k\in\mathbb{N}^*}$ de la façon suivante:

$$\begin{cases} x_1 & \text{point initial} \\ x_{k+1} = x_k + \alpha_k d_k & k \geq 1 \end{cases}.$$

Le pas $\alpha_k \in \mathbb{R}$ est déterminé par une recherche linéaire exacte ou inexacte.

Les directions d_k sont calculées de façon récurrente par les formules suivantes:

$$d_k = \begin{cases} -g_k & k = 1 \\ -g_k + \beta_k d_{k-1} & k \geq 2 \end{cases}$$

où $g_k = \nabla f(x_k)$ et β_k est un scalaire.

Les différentes valeurs attribuées à β_k définissent les différentes variantes de la méthode du gradient conjugué.

Parmi les β_k les plus connus, on a:

$$\begin{aligned}
\beta_k^{FR} &= \frac{\| g_k \|^2}{\| g_{k-1} \|^2}, \quad \beta_k^{CD} = -\frac{\| g_k \|^2}{g_{k-1}^T d_{k-1}}, \quad \beta_k^{DY} = \frac{\| g_k \|^2}{y_{k-1}^T d_{k-1}}, \\
\beta_k^{HS} &= \frac{g_k^T y_{k-1}}{y_{k-1}^T d_{k-1}}, \quad \beta_k^{PRP} = \frac{g_k^T y_{k-1}}{\| g_{k-1} \|^2}, \quad \beta_k^{LS} = -\frac{g_k^T y_{k-1}}{g_{k-1}^T d_{k-1}}, \\
\beta_k^{DHSDL} &= \frac{\| g_k \|^2 - \frac{\|g_k\|}{\|g_{k-1}\|} \mid g_k^T g_{k-1} \mid}{\mu \mid g_k^T d_{k-1} \mid + d_{k-1}^T y_{k-1}} - t\frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}}, \quad \mu > 1, \ t > 0, \\
\beta_k^{DLSDL} &= \frac{\| g_k \|^2 - \frac{\|g_k\|}{\|g_{k-1}\|} \mid g_k^T g_{k-1} \mid}{\mu \mid g_k^T d_{k-1} \mid - d_{k-1}^T g_{k-1}} - t\frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}}, \quad \mu > 1, \ t > 0,
\end{aligned}$$

où $y_{k-1} = g_k - g_{k-1}$, $s_{k-1} = x_k - x_{k-1}$ et $\| \, . \, \|$ désigne la norme euclidienne.

On considère deux nouvelles méthodes hybrides du gradient conjugué, parmi les plus récentes, la première est la méthode mixte (**MLSCD)** des deux variantes de Y. Zheng et B. Zheng (DHSDL et DLSDL), la seconde est la méthode (**MMDL)** de Liu-Story (LS) et celle de la descente conjuguée (CD).

On considère la méthode **MLSCD**, donnée avec la direction de recherche

$$d_k = \begin{cases} -g_1 & k = 1 \\ D(\beta_k^{LSCD}, g_k, d_{k-1}) & k \geq 2, \end{cases} ,$$

où

$$\beta_k^{LSCD} = \max\left\{0, \min\left\{\beta_k^{LS}, \beta_k^{CD}\right\}\right\},$$

et

$$D(\beta_k^{LSCD}, g_k, d_{k-1}) = -\left(1 + \beta_k^{LSCD} \frac{g_k^T d_{k-1}}{\|g_k\|^2}\right) g_k + \beta_k^{LSCD} d_{k-1}.$$

Algorithme MLSCD

Etape 0: Donner un point de départ x_1 et $\varepsilon > 0$.
Etape 1: Poser $k = 1$ et calculer $d_1 = -g_1$.
Etape 2: Si $\|g_k\| \leq \varepsilon$, **Stop**; sinon passer à **l'étape 3**.
Etape 3: Calculer le pas $\alpha_k \in]0, 1]$ (par Armijo ou Wolfe ou Backtracking).
Etape 4: Calculer $x_{k+1} = x_k + \alpha_k d_k$.
Etape 5: Calculer g_{k+1}, $y_k = g_{k+1} - g_k$ et passez à **l'étape 6.**
Etape 6: Calculer
$\beta_{k+1}^{LS} = -\frac{y_k^T g_{k+1}}{g_k^T d_k}$, $\beta_{k+1}^{CD} = -\frac{\|g_{k+1}\|^2}{g_k^T d_k}$,
$\beta_{k+1}^{LSCD} = \max\left\{0, \min\left\{\beta_{k+1}^{LS}, \beta_{k+1}^{CD}\right\}\right\}$.
Etape 7: Calculer la direction de recherche $d_{k+1} = D(\beta_{k+1}^{LSCD}, g_{k+1}, d_k)$.
Etape 8: Poser $k = k + 1$ et passer à **l'étape 2**.

Pour la seconde méthode hybride **MMDL,** on a la diretion de recherche d_k est donnée comme suit :

$$d_k = \begin{cases} -g_1 & k=1 \\ D(\beta_k^{MMDL}, g_k, d_{k-1}) & k \geq 2 \end{cases},$$

où

$$\beta_k^{MMDL} = \max\left\{0, \min\left\{\beta_k^{DHSDL}, \beta_k^{DLSDL}\right\}\right\},$$

et

$$D(\beta_k^{MMDL}, g_k, d_{k-1}) = -\left(1 + \beta_k^{MMDL} \frac{g_k^T d_{k-1}}{\|g_k\|^2}\right) g_k + \beta_k^{MMDL} d_{k-1}.$$

Algorithme MMDL

Etape 0: Donner un point de départ x_1 et les paramètres $\varepsilon > 0$, $\mu > 1$.
Etape 1: Poser $k = 1$ et calculer $d_1 = -g_1$.
Etape 2: Si $\|g_k\| \leq \varepsilon$ **Stop**; sinon passer à **l'étape 3**.
Etape 3: Calculer le pas $\alpha_k \in]0,1]$ (par Armijo ou Wolfe ou Backtracking)
Etape 4: Calculer $x_{k+1} = x_k + \alpha_k d_k$.
Etape 5: Calculer g_{k+1}, $y_k = g_{k+1} - g_k$, $s_k = x_{k+1} - x_k$ et passer à **l'étape 6.**
Etape 6: Calculer

$$\beta_{k+1}^{DHSDL} = \frac{\|g_{k+1}\|^2 - \frac{\|g_{k+1}\|}{\|g_k\|}\left|g_{k+1}^T g_k\right|}{\mu\left|g_{k+1}^T d_k\right| + d_k^T y_k} - \alpha_k \frac{g_{k+1}^T s_k}{d_k^T y_k},$$

$$\beta_{k+1}^{DLSDL} = \frac{\|g_{k+1}\|^2 - \frac{\|g_{k+1}\|}{\|g_k\|}\left|g_{k+1}^T g_k\right|}{\mu\left|g_{k+1}^T d_k\right| + d_k^T g_k} - \alpha_k \frac{g_{k+1}^T s_k}{d_k^T y_k},$$

$$\beta_{k+1}^{MMDL} = \max\left\{0, \min\left\{\beta_{k+1}^{DHSDL}, \beta_{k+1}^{DLSDL}\right\}\right\}.$$

Etape 7: Calculer la direction de recherche $d_{k+1} = D(\beta_{k+1}^{MMDL}, g_{k+1}, d_k)$.
Etape 8: Poser $k = k + 1$ et passer à **l'étape 2**.

Dans ce travail, on a fait une étude comparative numérique entre ces deux méthodes hybrides du gradient conjugué (**MMDL** et **MLSCD),** en utilisant différentes règles de rcherche linéaire inexacte, à savoir celle d'Armijo, de Wolfe et de Backtracking, pour calculer le pas de déplacement α_k. Les tests numériques ont été effectués sur plusieurs fonctions tests et pour différentes dimensions (n).

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Existence Result of Positive Solution for a Degenerate parabolic System via a Method of Upper and Lower Solutions

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1 Abstract

The aim of this paper is to prove the existence of positive maximal and minimal solutions for a class of degenerate parabolic reaction diffusion systems,including the uniqueness of the positive solution. To answer these questions, we use a technique based on the method of upper and lower solutions.
Keywords : reaction diffusion systems, degenerate parabolic systems, upper and lower solutions.
MSC 2010 : 35J62, 35J70, 35K57.

2 Introduction

Degenerate quasilinear parabolic and elliptic equations have received extensive attentions during the past several decades and many topics in the mathematical analysis are well developed and applied to various fields of applied sciences, especially in ecology as in this work.
In this paper, we consider a coupled system of arbitrary number of quasilinear parabolic equations in a bounded domain with Dirichlet boundary condition where the domain is assumed to have the outside sphere property without the usual smoothness condition. The system of equations under consideration is given by

$$\begin{cases} \frac{\partial u_i}{\partial t} - \operatorname{div}\left(D_i\left(u_i\right)\nabla u_i\right) + b_i.\left(D_i\left(u_i\right)\nabla u_i\right) = f_i\left(t,x,u\right) \quad , \quad (t,x)\in Q_T \\ u_i\left(t,x\right) = g_i\left(t,x\right) \quad , \quad (t,x)\in S_T, \\ u_i\left(0,x\right) = h_i\left(x\right) \quad , \quad x\in\Omega, \qquad i=1,...,N \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ and $Q_T = [0,T]\times\Omega$ and $S_T = [0,T]\times\partial\Omega$. $D_i\left(u_i\right), f_i(t,x,u)$ and $g_i(t,x)$, $h_i\left(x\right)$ are prescribed functions satisfying the following hypotheses :

(H_1) $f_i\left(t,x,.\right)\in C^{\frac{\alpha}{2},\alpha}\left(\bar{\Omega}\right)$, $f_i\left(t,x,0\right)\geq 0$ in Q_T and $g_i\left(t,x\right)\in C^{\alpha}\left(S_T\right)$.

(H_2) $D_i\left(u\right)\in C^2\left([0,M_1]\right)$, $D_i\left(u\right)>0$ in $(0,\ M_i]$, and $D_i\left(0\right)\geq 0$ with $M_i = \|\tilde{u}_i\|_{C\left(\bar{\Omega}\right)}$.

(H_3) $f_i(.,\mathbf{u}) \in C^1(S^*)$, and

$$\frac{\partial f_i}{\partial u_j} \geq 0 \ , \ for \ j \neq i \ , \ \mathbf{u} \in S^*$$

(H_4) $g_i(t,x) \geqslant 0$ on $S_T, \psi_i(x) > 0$ in Ω, and $g_i(0,x) = h_i(x)$ on $\partial\Omega$.

(H_5) There exists a constant $\delta_0 > 0$ such that for any $x_0 \in \partial\Omega$ there exists a ball $\mathbf{K}$ outside of Ω with radius $r \geq \delta_0$ such that $\mathbf{K} \cap \bar{\Omega} = \{x_0\}$.

we denote $C^{\alpha}(\Omega)$ to the space of Hölder continuous functions in Ω.
In the above system, we further assume $D_1(0) = 0$ or $D_2(0) = 0$.
Let $\gamma_1(x)$ and $\gamma_2(x)$ be smooth positive functions satisfying

$$c_i(x) \geq \max\left\{-\frac{\partial f}{\partial u}(t,x,\ \mathbf{u})\ ;\ \mathbf{u} \in S^*\right\}; \tag{2}$$

$i = 1, ..., N$
First, we have to clarify in which sense we want to solve our problem :

Definition 1 *A pair of functions* $\tilde{\mathbf{u}} = (\tilde{u}_1, \ldots, \tilde{u}_N), \hat{\mathbf{u}} = (\hat{u}_1, \ldots, \hat{u}_N)$ *in* $\mathcal{C}(\bar{Q}_T) \cap \mathcal{C}^2(Q_T)$ *are called ordered upper and lower solutions of (1.1) if* $\hat{\mathbf{u}} \leqslant \tilde{\mathbf{u}}$ *and if*

$$\begin{cases} \frac{\partial \hat{u}_i}{\partial t} - \operatorname{div}(D_i(\hat{u}_i)\nabla\hat{u}_i) + \mathbf{b}_i \cdot (D_i(\hat{u}_i)\nabla\hat{u}_i) \leqslant f_i(t,x,\hat{\mathbf{u}}) & in\ Q_T \\ \hat{u}_i(t,x) \leqslant g_i(t,x) \quad on\ S_T \\ \hat{u}_i(0,x) \leqslant h_i(x) \quad in\ \Omega, i = 1, \ldots, N \end{cases}$$

and $\tilde{u}$ *satisfies the above inequalities in reversed order. It is obvious that every solution of (1.1) is an upper solution as well as a lower solution. For a given pair of ordered upper and lower solutions* $\tilde{\mathbf{u}}, \hat{\mathbf{u}}$,

For a given pair of ordered upper and lower solutions $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$, we define

$$S_i \equiv \{u_i \in C^{\alpha}(Q_T) \cap C(\bar{Q}_T)\,; \hat{u}_i \leqslant u_i \leqslant \tilde{u}_i\} \quad (i = 1, \ldots, N),$$
$$S = \{\mathbf{u} \in \mathcal{C}^{\alpha}(Q_T) \cap \mathcal{C}(\bar{Q}_T) : \hat{\mathbf{u}} \leqslant \mathbf{u} \leqslant \tilde{\mathbf{u}}\}.$$

Now, we assume the following assumptions :

(H_1) $f_i(t,x,.) \in C^{\frac{\alpha}{2},\alpha}(\bar{\Omega})$, $f_i(t,x,0) \geq 0$ in Q_T and $g_i(t,x) \in C^{\alpha}(S_T)$.

(H_2) $D_i(u) \in C^2([0, M_1])$, $D_i(u) > 0$ in $(0,\ M_i]$, and $D_i(0) \geq 0$ with $M_i = \|\tilde{u}_i\|_{C(\bar{\Omega})}$.

(H_3) $f_i(x,.) \in C^{\alpha}(\bar{\Omega})$, $f_i(.,u) \in C^1(S^*)$, and

$$\frac{\partial f_i}{\partial u_j} \geq 0 \ , \ for \ j \neq i \ , \ u \in S^*$$

(H_4) $g_i(t,x) \geqslant 0$ on $S_T, \psi_i(x) > 0$ in Ω, and $g_i(0,x) = h_i(x)$ on $\partial\Omega$.

(H_5) There exists a constant $\delta_0 > 0$ such that for any $x_0 \in \partial\Omega$ there exists a ball $\mathbf{K}$ outside of Ω with radius $r \geq \delta_0$ such that $\mathbf{K} \cap \bar{\Omega} = \{x_0\}$.

2.1 The main result

Now, we can state the main result of this paper :

Theorem 1 *Let* $\tilde{\mathbf{u}}_s$, $\hat{\mathbf{u}}_s$ *be ordered positive upper and lower solutions of* (1), *and let hypotheses* $(H_1) - (H_5)$ *hold. Then problem* (1) *has a unique positive solution* $\mathbf{u}_s^*$ *that satisfies* $\hat{\mathbf{u}}_s \leq \mathbf{u}_s^* \leq \tilde{\mathbf{u}}_s$. *Moreover, the sequences* $\{\underline{\mathbf{u}}^m\}$, $\{\overline{\mathbf{u}}^m\}$ *with* $\underline{\mathbf{u}}(0) = \hat{\mathbf{u}}_s$ *and* $\overline{\mathbf{u}}(0) = \tilde{\mathbf{u}}_s$ `converge monotonically to` $\mathbf{u}_s^*$ *and satisfy the relation*

$$\hat{\mathbf{u}}_s \leq \underline{\mathbf{u}}_s^{(m)} \leq \underline{\mathbf{u}}_s^{(m+1)} \leq \overline{\mathbf{u}}_s^{(m+1)} \leq \overline{\mathbf{u}}_s^{(m)} \leq \tilde{\mathbf{u}}_s \ , \ \textit{for all } m \geq 1$$

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Existence and Uniqueness for a System of klein-Gordon equations

LATIOUI NAAIMA. GUESMIA AMAR

ABSTRACT

In our paper we study the weak existence of a non linear hyperbolic coupled system of Klein-Gordon equations with memory and source terms by using the Faedo-Galerkin method thechniques and compactness result, we have demonstrated the uniqueness of the solution by using the classical technique .

KEYWORDS AND PHRASES. Klein-Gordon system , Faedo-Galerkin method, source term.

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EXISTENCE DU HYPERCHAOS DANS UN NOUVEAU SYSTÈME DE RABINOVICH D'ORDRE FRACTIONNAIRE AVEC UN SEUL TERME NON LINÉAIRE

SMAIL KAOUACHE

Résumé. Dans ce travail de cette communication, on va proposer un nouveau système hyperchaotique fractionnaire généré à partir d'une petite modification du système classique de Rabinovich. Malgré que notre système est d'ordre fractionnaire et de plus admet un seul élément non linéaire, on va monter que ce système peut exhiber des comportements hyperchaotiques. Nous abordons les propriétés dynamiques ainsi que le problème de la stabilité asymptotique de ce système. Cette stabilité est réalisée via un contrôleur continu. L'utilisation de la méthode fractionnaire de Lyapounov ainsi qu'une propriété importante de la dérivée fractionnaire de Caputo pour les systèmes fractionnaires nous permet de conclure sur la convergence asymptotique des états du système proposé. Des simulations numériques sont illustrés pour tester l'efficacité du système proposé.

2010 MSC. 34A34, 37B55, 93C55, 93D05 .

Mots Clés. Système de Rabinovich, Dérivée fractionnaire, Contrôle continu, Systèmes hyperchaotiques fractionnaires.

1. Quelques outils de dérivation fractionnaire au sens de Caputo

L'expression mathématique de la dérivée fractionnaire au sens de caputo est donnée par :

$$ {}_{a}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{1} $$

où Γ représente la fonction de Gamma et $\alpha \in (0,1)$ est l'ordre de dérivation.

Theorem 1.1. [3] *Considérons le système non linéaire fractionnaire décrit par le modèle suivant;*

$$ \begin{cases} D^{\alpha}x = f(x), \\ x(0) = x_0, \end{cases} \tag{2} $$

où $x \in \mathbb{R}^n$, $0 < \alpha < 1$ et $f \in \mathbb{R}^n$ une fonction non linéaire continue.
Soient $\lambda_1, \lambda_2, ..., \lambda_n$ les valeurs propres de la matrice jacobienne $\frac{\partial f}{\partial x}$ associée à f au point d'équilibre.
Alors, le système (2) est asymptotiquement stable, si et seulement si :

$$ |\arg(\lambda_i)| > \alpha\frac{\pi}{2}, \ \textit{pour tout}\ i = 1, 2, ..., n. \tag{3} $$

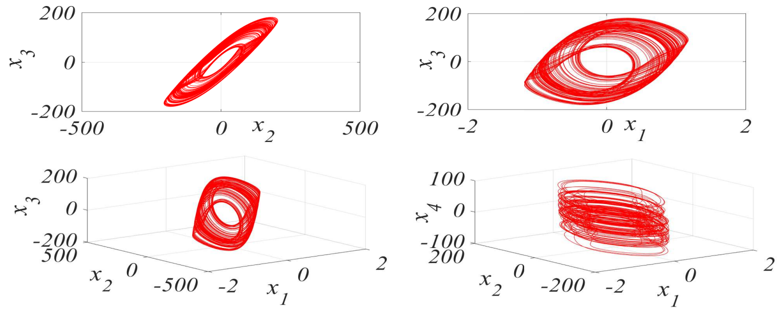

FIGURE 1. Projections de portrait de phase du système (5).

Lemma 1.2. [5] *Soit $x \in \mathbb{R}^n$, une fonction dérivable au sens de Caputo. On a alors, pour tout $\alpha \in (0,1)$,*

$$\frac{1}{2}D^\alpha x^T(t)x(t) \leq x^T(t)D^\alpha x(t). \tag{4}$$

Theorem 1.3. [4] *Lorsqu'il existe une fonction de Lyapounov positive $V(x)$, telle que $D^\alpha(V(x)) < 0$, pour tout $t \geq t_0$, alors la solution de système (2) est asymptotiquement stable.*

2. EXISTENCE DU HYPERCHOS D'UN NOUVEAU SYSTÈME FRACTIONNAIRE DE RABINOVICH

Notre nouveau système hyperchaotique est décrit par le modèle fractionnaire suivant

$$\begin{cases} D^\alpha x_1 = -a_1x_1 + a_2x_2, \\ D^\alpha x_2 = a_2(x_1 - x_4) - a_3(x_2 - x_3) + x_3 - x_1^2x_2, \\ D^\alpha x_3 = -a_4(x_3 - x_2), \\ D^\alpha x_4 = -a_5x_2, \end{cases} \tag{5}$$

où x_1, x_2, x_3, x_4 sont les variables d'état, a_1, a_2, a_3, a_4, a_5 sont des constantes réelles, D^α est l'opérateur de dérivation au sens de Caputo, et α est l'ordre de dérivation compris entre 0 et 1. Lorsque $\alpha = 0.98$ et les paramètres du système sont donnés par

$$a_1 = 1,\ a_2 = 0.01,\ a_3 = 1.7,\ a_4 = -2.5 \text{ et } a_5 = 0.03, \tag{6}$$

le système proposé peut exhiber un comportement hyperchaotique.
Les projections de portrait de phase sur les plans : x_1-x_3, x_2-x_3, $x_1-x_2-x_3$ et $x_1-x_2-x_4$ sont représentés dans la Figure 1. Les exposants fractionnaires de Lyapounov du système (5) sont donnés par :

$$L_1 = 0.11,\ L_2 = 0.08,\ L_3 = 0 \text{ et } L_4 = -0.63. \tag{7}$$

Ce qui assure que le système est bien hyperchaotique.
De plus, $\sum_{i=1}^4 L_i = -0.44 < 0$, ce qui montre une fois encore que le système est bien dissipatif, et par conséquent, le volume du système va diminuer de la valeur V_0 vers 0. Cela signifie que toutes les trajectoires de ce système convergent finalement vers un attracteur, quand $t \to +\infty$.

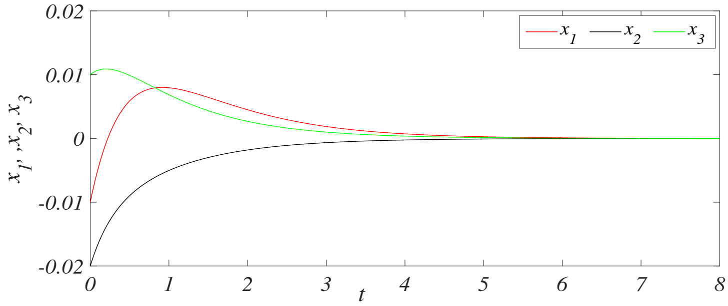

FIGURE 2. Courbes des états du système contrôlé.

3. Etude de la stabilité asymptotique

La principale motivation de cette partie est de construire un contrôle actif pour assurer la stabilité du système proposé.

3.1. **Résultats théoriques.** Pour quantifier notre objectif, considérons le système contrôlé suivant

$$(8) \qquad \begin{cases} D^{\alpha}x_1 = -a_1x_1 + a_2x_2 + u_1, \\ D^{\alpha}x_2 = a_2(x_1 - x_4) - a_3(x_2 - x_3) + x_3 - x_1^2x_2 + u_2, \\ D^{\alpha}x_3 = -a_4(x_3 - x_2) + u_3, \\ D^{\alpha}x_4 = -a_5x_1 + u_4, \end{cases}$$

où $0 < \alpha < 1$ et u_1, u_2, u_3, u_4 sont des paramètres de contrôle.
Ce système peut être représenter sous forme matricielle comme suit

$$(9) \qquad D^{\alpha}x = Px + f(x) + u,$$

où $x = (x_1, x_2, x_3, x_4)^T$, $u = (u_1, u_2, u_3, u_4)^T$, P et $f : \mathbb{R}^4 \to \mathbb{R}^4$ sont respectivement la partie linéaire et la partie non linéaire du système(9).

Theorem 3.1. *Supposons que le contrôle continu u est structuré de la façons suivant*

$$(10) \qquad u = -f(x) + Cx,$$

où $C \in \mathbb{R}^{4\times4}$ est la matrice de gain à déterminé.
Si la matrice C est sélectionnée de tel sorte que la matrice $P + C$ est définie négative, notre système proposé converge asymptotiquement vers zéro.

3.2. **Résultats de simulation.** Dans les simulations numériques, la méthode Adams-Bashforth-Moulton est utilisée pour résoudre notre système fractionnaire. Avec des choix particuliers de C et de u, les états variables du système convergent asymptotiquement vers zéro comme nous voyons dans la Figure 2.

4. Conclusion

Dans ce travail de cette communication, un nouveau système hyperchaotique d'ordre fractionnaire ayant un seul terme non linéaire a été proposé. Le problème de la stabilité fractionnaire de ce système a été également étudié. Cette stabilité a été réalisée via un contrôleur continu. Une analyse de Lyapounov ainsi qu'une propriété importante de la dérivée fractionnaire de

Caputo pour les systèmes fractionnaires ont été effectuée pour conclure sur la stabilité ainsi que la convergence des états du système. Des simulations numériques ont été illustré pour tester l'efficacité du système proposé.

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FEEDBACK BOUNDARY STABILIZATION OF THE SCHRÖDINGER EQUATION WITH INTERIOR DELAY

WASSILA GHECHAM, SALAH-EDDINE REBIAI, AND FATIMA ZOHRA SIDI ALI

ABSTRACT. In [1] Ammari et al established, under Lions geometric condition, an exponential stability result for the wave equation with an interior delay term and a Neumann boundary feedback. Boundary stabilization problems for the undelayed Schrödinger equation were considered in [2] and [3]. In [4], stability problems for the Schrödinger equation with a delay term in the boundary or internal feedbaks were investigated. Our aim in this paper is to study the boundary stabilization problem for the Schrödinger equation with an interior time delay. Under suitable assumptions, we prove exponential stability of the solution. This result is obtained by using multiplier techniques and by introducing a suitable lyapunov functional.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 93D15; 35J10.

KEYWORDS AND PHRASES. Schrödinger equation, interior delay, boundary stabilization.

1. DEFINE THE PROBLEM

Let Ω be an open bounded domain of $\mathbb{R}^n$ with boundary Γ of class C^2 which consists of two non-empty parts Γ_1 and Γ_2 such that, $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$. In Ω, we consider the following Schrödinger equation with interior delay term and dissipative boundary feedback:

(1)
$$\begin{cases} u_t(x,t) - i\Delta u(x,t) + \alpha u(x,t-\tau) = 0 & \text{in } \Omega \times (0;+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = -\beta u_t(x,t) & \text{on } \Gamma_2 \times (0,+\infty), \\ u(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega \times (0,\tau), \end{cases}$$

where

- u_0 and f_0 are the initial data which belong to a suitable spaces.
- $\frac{\partial}{\partial \nu}$ is the normal derivative.
- $\tau > 0$ is the time delay.
- α and β are a positive constants.

In this work, we are interested in studying the well-posedness and the stability problems of the Schrödinger equation with interior delay and a dissipative boundary feedback as described in (1).

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FRACTIONAL DIFFERENTIAL EQUATIONS OF CAPUTO-HADAMARD TYPE AND NUMERICAL SOLUTIONS

KAOUTHER BOUCHAMA, ABDELKRIM MERZOUGUI, AND YACINE ARIOUA

ABSTRACT. This paper is concerned with a numerical method for solving generalized fractional differential equation of Caputo-Hadamard derivative. A corresponding discretization technique is proposed. Numerical solutions are obtained and convergence of numerical formula is discussed. The convergence speed arrives at $O(h^{1-\alpha})$. Numerical examples are given to test the accuracy.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 65C20, 34A08, 26A33.

KEYWORDS AND PHRASES. Numerical method, Fractional differential equations, Caputo-Hadamard fractional derivative.

1. DEFINE THE PROBLEM

In this paper, we consider a numerical technique for the fractional differential equation of Caputo–Hadamard type:

$$\begin{cases} {}^{CH}\mathcal{D}_{a^+}^{\alpha} u\,(t) + cu(t) = f(t), \; 0 < a \leq t \leq b < \infty \\ u(a) = u_a \end{cases} \tag{1}$$

Where ${}^{CH}\mathcal{D}^{\alpha}$ denotes the Caputo-Hadamard fractional derivative operator of order $\alpha \in (0,1]$.The discrete implicit Euler formula is applied to obtain an approximate sequence for (1). In the first case, the equidistance partition is used to obtain a discrete version of the Caputo-Hadamard derivative, then the numerical formula and the numerically solve of the fractional differential equation are obtained.

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FREE SURFACE FLOWS OVER A TWO OBSTACLES BY USING SERIES METHOD

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Abstract. Free-surface two-dimensional flows past a successive triangular obstacles is considered. We suppose that the fluid is incompressible and non-viscous. The flow is assumed to be steady and irrotational. The gravity and the surface tension are included in the free surface condition. The problem is solved numerically by employing series-truncation method. The numerical solutions exist for various values of the Weber number and the Froude number. When the surface tension tends to zero, It is shown that there are solutions for which the flow is supercritical and sub-critical both upstream and downstream. The free surface profiles are plotted for different sizes of successive triangles

2010 Mathematics Subject Classification. 35B40, 76B07, 76M45.

Keywords and phrases. Free surface flow; potential flow; Weber number; surface tension; Froude number.

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GLOBAL EXISTENCE OF WEAK SOLUTIONS FOR 2×2 PARABOLIC FULL REACTION-DIFFUSION SYSTEMS APPLIED TO A CLIMATE MODEL

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Abstract. This work concerns the global existence in time of weak solutions for the strongly coupled reaction-diffusion system with a full matrix of diffusion coeffcients for which two main properties hold: the positivity of the solutions and the total mass of the components are preserved with time. Moreover we suppose that the non-linearities have critical growth with respect to the gradient. The technique we use here in order to prove global existence is in the same spirit of the method developed by Boccardo, Murat, and Puel for a single equation.

Our investigation applied for a wide class of the nonlinear terms f and g.

1. Define the problem

The modeling and the mathematical analysis of parabolic systems, in particular, reaction diffusion systems, has been the subject of in-depth studies of several mathematicians in recent years, as they appear in the modeling of a large variety of phenomena, not only in biology and chemistry, but also in engineering, economics and ecology, such as gas dynamics, fusion processes, cellular processes, disease propagation, industrial processes , catalytic transport of contaminants in the environment, population dynamics, flame spread and others.. For systematic expositions of some aspects of the theory, numerous applications, and a comprehensive list of literature on this subject we refer to [15, 16, 8, 10, 2].

We are interested in global existence in time of solutions to the reaction-diffusion systems of the form

$$\frac{\partial u}{\partial t} - a\Delta u - b\Delta v = f\left(t, x, u, v, \nabla u, \nabla v\right), \text{ in } Q_T, \tag{1.1}$$

$$\frac{\partial v}{\partial t} - c\Delta u - a\Delta v = g\left(t, x, u, v, \nabla u, \nabla v\right), \text{ in } Q_T, \tag{1.2}$$

with the following boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \text{ or } u = v = 0, \text{ in } \Sigma_T, \tag{1.3}$$

supplemented with the initial conditions

$$u\left(0, x\right) = u_0\left(x\right), \; v\left(0, x\right) = v_0\left(x\right), \text{ in } \Omega, \tag{1.4}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $Q_T =]0, T[\times \Omega$, $\Sigma_T =]0, T[\times \partial\Omega$, $T > 0$, and Δ denotes the Laplacian operator on

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Key words and phrases. Global solution, semigroups, local solution, reaction-diffusion systems.

$L^1(\Omega)$ with respect to the x variable with homogeneous Neumann or Dirichlet boundary conditions. The diffusion coefficients a, b and c are positive constants satisfying the condition $2a > (b+c)$ which reects the parabolicity of the system. The system (1.1)-(1.2) may be regarded as a perturbation of the simple and trivial case where $b = c = 0$, for which nonnegative solutions exist globally in time.
Always in this case with homogeneous Neumann boundary conditions but when the coeffcient of $-\Delta u$ in the first equation is different of the one of $-\Delta v$ in the second one (diagonal case), Alikakos [14] established global existence and L^∞-bounds of solutions for positive initial data in the case

$$f(u,v) = -g(u,v) = -uv^\sigma$$

where $1 < \sigma < \dfrac{n+2}{n}$. Masuda [5] showed that solutions to this system exist globally for every $\sigma > 1$ and converge to a constant vector as $t \to +\infty$. Haraux and Youkana [?] have generalized the method of Masuda to handle nonlinearities $f(u,v) = g(u,v) = -u\Psi(v)$ that are from a particular case of our one. In [3], Moumeni and Barrouk obtained a global existence result. By combining the compact semigroup methods and some L^1 estimates, we show that global solutions exist for a large class of the functions f and g. Recently Kouachi and Youkana [21] have generalized the method of Haraux and Youkana by adding $-c\Delta v$ to the left-hand side of the diagonal case and by taking nonlinearities $f(u,v)$ of a weak exponential growth. Kanel and Kirane [9] have proved global existence, in the case $g(u,v) = -f(u,v) = -uv^n$ and n is an odd integer, under an embarrassing condition which can be written in our case as

$$|b-c| < C_p,$$

where C_p contains a constant from an estimate of Solonnikov. Recently they ameliorate their results in [?] to obtain global existence under the conditions

$$b < \left(\frac{a^2}{a^2+c^2}\right) c$$

and

$$|F(v)| \leq C_F \left(1 + |v|^{1+\alpha}\right)$$

where α and C_F are positive constants with $\alpha < 1$ suffciently small and $g(u,v) = -f(u,v) = -uF(v)$. All techniques used by authors cited above showed their limitations because some are based on the embedding theorem of Sobolev as Alikakos [14], Hollis, *et all* [22], ... another as Kanel and Kirane [9] use a properties of the Neumann function for the heat equation for which one of it's restriction the coeffcient of $-\Delta u$ in equation (1.1) must be larger than the one of $-\Delta v$ in equation (1.2) whereas it isn't the case of problem (1.1)-(1.4).
Moumeni and Barrouk [4] has proved the global existence of solutions for two-component reaction-diffusion systems for the same system with homogeneous Dirichlet boundary conditions.
On the same direction, Kouachi [20] has proved the global existence of solutions for two-component reaction-diffusion systems with a general full matrix of diffusion coeffcients, nonhomogeneous boundary conditions and polynomial growth conditions on the nonlinear terms.

In this present article we consider the problem (1.1)-(1.4) by using a homogeneous Neumann or Dirichlet boundary conditions we establish a global existence result of the solution.
The components $u(t,x)$ and $v(t,x)$ represent either chemical concentrations or biological population densities and system (1.1)-(1.2) is a mathematical model describing various chemical and biological phenomena (see Cussler [6], Garcia, Ybarra and Clavin [17], Groot and Mazur [23]).

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GALERKIN APPROXIMATION OF THE DIFFUSION-REACTION EQUATION BY CUBIC B-SPLINES

NOURIA ARAR

Abstract. This work is devoted to the development of a Galerkin-type approximation of the solution of the diffusion-reaction equation, using cubic B-Spline functions and a Runge Kutta of order 4 finite difference scheme. Examples are used to validate the proposed approximation. The numerical results obtained show the effectiveness of the procedure.

2010 Mathematics Subject Classification. 65D07, 65N30, 65N22, 65N06.

Keywords and phrases. diffusion-reaction equation, Finite differences, Galerkin method, Finite elements, cubic B-splines.

1. Define the problem

We consider the diffusion-reaction problem with homogeneous boundary conditions.

$$(1)\quad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \alpha \dfrac{\partial^2 u}{\partial x^2}(t,x) + \beta u(t,x) = f(t,x) & -1 < x < 1;\ \ t > 0 \\ u(t,-1) = u(t,1) = 0 \\ u(0,x) = u_0 = g(x) \end{cases}$$

where $g(x)$ a given initial condition.
In this study, we focus on the case where the reaction and diffusion coefficients are scalars. Let $\alpha,\ \beta \in \mathbb{R}$.

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GLOBAL STABILITY OF COVID-19 EPIDEMIC MODEL

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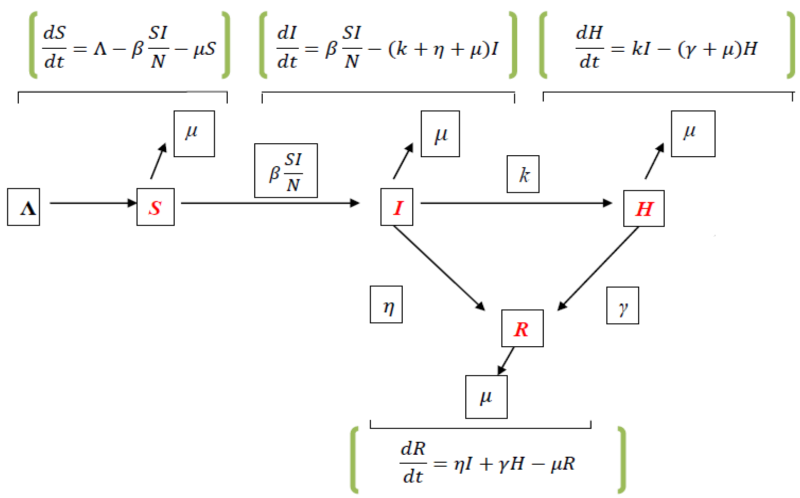
Abstract. In this study, a system of first order ordinary differential equations is used to analyse the dynamics of COVID-19 disease via a mathematical model proposed.The global stability analysis is conducted for the extended model by suitable Lyapunov function, in which either susceptible or infective populations are diffusive. The stability of the disease is dependent on both transmission rate of the disease and the progression rate of the infectious state to isolated or hospitalized state. The number R_0 can be played role in determining whether the disease will extinct or persist, if $R_0 < 1$, then the disease-free equilibrium is globally asymptotically stable and unstable when $R_0 > 1$.

Keywords: Stability local, Stability global, Equilibriums points, Lyapunov function.

The mathematical model of the transmission of COVID-19 is described as follows:



In the model, total population $N(t)$ is divided into four classes: Susceptible: S, Infected: I, Hospitalized: $H$and Recovered: R. So, $N(t) = S(t) + I(t) + H(t) + R(t)$.

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History-dependent hyperbolic variational inequalities with applications to contact mechanics

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March 10, 2021

Abstract

The aim of this work is to study an abstract hyberbolic variational inequalities with a history dependent operator. a result on its solvability is proved by applying the time-discretization technique and monotone operators theory. We illustrate the abstract results by an application to dynamic contact frictional problem for viscoelastic materials.

2010 Mathematics Subject Classification. 5K15, 49J40, 70G75, 70F40

Keywords: Variational inequalities, hyperbolic, time-discretization technique.

1 Definition of problem:

In this work we establish the existence of solution to hyperbolic variational inequalities with a history dependent operator arising in dynamic viscoelastic frictional contact problem. By a time-discretization technique and monotone operators theory, the inqualities are solved in the form of evolutionary inclusions.

Here, V is a Banach space of admissible displacements, and we introduce A and B are operators related to the viscoelastic constitutive law, $\mathcal{C}$ represents a history-dependent operator φ is a convex functional related to contact boundary conditions, and J^0 denotes the generalized gradient of a locally Lipschitz function J. The function f represents the given body forces and surface traction, and u_0, u_1 represents the initial displacement and velocity, respectively.

2 Existence of solution

In this section , we consider an evolution of triple spaces $V \subset H \subset V^*$, where V is a strictly convex, reflexive and separable Banach space, H is a separable Hilbert space. For $0 < T < +\infty$, we consider the standard Bochner-Lebesgue function spaces $\mathcal{V} = L^2(0, T; V)$ and $\mathcal{W} = \{v \in \mathcal{V} | v' \in \mathcal{V}^*\}$, where $v' = \partial v/\partial t$ is the time derivative in the sense of vector-valued distributions. By the reflexivity of V we have both $\mathcal{V}$ and its dual $\mathcal{V}^* = L^2(0, T; V^*)$ are reflexive Banach spaces.

Let A, B are operators related to the viscoelastic constitutive law, we have X is Banach space, a functionals $\varphi : [0, T] \times X \to \mathbb{R}$, $J : [0, T] \times X \to \mathbb{R}$ and $f \in \mathcal{V}^*, u_0, u_1 \in V$, we consider the hyperbolic variational inequality of finding an element $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$

with some hypotheses H(A),H(B), H(φ),H(J), H($\mathcal{C}$), H(L) and H(f).

$$\begin{cases} \langle u''(t) + \psi(t) + Bu(t) + (\mathcal{C}u)(t) - f(t), v - u'(t)\rangle + \varphi(t, v) - \varphi(t, u'(t)) \\ +J^0(t, Lu'(t); Lv - Lu'(t) \geq 0 \quad \text{for all} \quad v \in \mathcal{V}, a.e. t \in (0, T) \\ u(0) = u_0, \qquad u'(0) = u_1 \qquad \psi(t) \in A(t, u'(t)) \end{cases} \tag{1}$$

$$(\mathcal{C}u)(t) = E\left(\int_0^t q(t,s)u(s)ds + \alpha\right) \quad \text{for } t \in [0, T]$$

We make the following hypotheses.

H(A): $A : [0, T] \times V \to 2^{V^*}$ is multivalued operator such that
(a) $A(., v) : [0, T] \times V \to 2^{V^*}$ is measurable for all $v \in V$;
(b) $A(t, .)$ is pseudomonotone for a.e. $t \in [0, T]$; there exist $a_1 \in L^2(0, T)$ and a constant $c_1 > 0$ such that for all $v \in V$

$$\|\psi(t)\|_{V^*} \leq a_1(t) + c_1\|\psi(t)\|_V, \ \ \forall \ \ \psi(t) \in A(t, v) \ \text{ and a.e } t \in [0, T]$$

(c) there exist $a_2 \in L^1_+(0, T)$ and $c_2 > 0$ such that for all$v \in V$

$$\langle\psi(t), v\rangle \geq c_2\|v\|^2_V - a_2(t), \ \ \forall \ \ \psi(t) \in A(t, v) \ \text{ and } a.e \ t \in [0, T]$$

H(B) $B : V \to V^*$ is linear, bounded, symmetric and monotone, i.e.,
(a) $B \in \mathcal{L}(V, V^*)$ and $\forall \ v \in V$, $\|B(v)\|_{V^*} \leq c_3\|v\|_V$ with $c_3 > 0$;
(b) $\langle B(u), v\rangle = \langle B(v), u\rangle \geq 0$, $\forall u, v \in V$

H(φ) $\varphi : [0, T] \times X \to \mathbb{R}$ is such that
(a) $\varphi(., u)$ is measurable for all $u \in X$ and $\varphi(u, .)$ is proper, convex and lower semicontinuous for a.e. $t \in [0, T]$
(b) there exist a function $a_3 \in L^2(0, T)$ and $c_4 > 0$ such that

$$\|\eta\|_{X^*} \leq a_3(t) + c_4\|v\|_X, \ \forall v \in M, \ \eta \in \partial\varphi(t, v) \ \ a.e.t \in [0, T]$$

(c) the mapping $\partial\varphi(., .)$ is upper semicontinuous endowed with the weak topology from $X \times X$ to X^*.

H(J) $J : [0, T] \times X \longrightarrow \mathbb{R}$ is such that
(a) $J(., v)$ is measurable on $[0, T]$ for all $v \in X$
(b) $J(t, .)$ is locally Lipschitz on X for a.e $t \in [0, T]$.
(c) The growth condition holds $\|\partial J(t, v)\|_{X^*} \leq c_0(t)+c_1(t)$ for all $v \in X$ a.e $t \in [0, T]$ with $c_0 \in L^2(0, T)$ and $c_0 \geq 0$, $c_1 \geq 0$

H(L) the operator $L : V \to X$ is linear and compact with its adjoint operator L^*.

H($\mathcal{C}$) $E : V \longrightarrow V^*$, $\alpha \in V$ and $q : [0, T] \times [0, T] \longmapsto (\mathcal{V}, \mathcal{V})$
H(E) $E \in \mathcal{L}(V, V^*)$.
H(q) The function $q \in C([0, T] \times [0, T], \mathcal{L}(V, V))$ is lipschitz continuous with respect to the first variabl, i.e ther exists $L_q > 0$ such that $\|q(t_1, s) - q(t_2, s)\| \leq L_q|t_1 - t_2|$ for all $t_1, t_2, s \in [0, T]$

H(f) $f \in L^2(0,T;V^*)$ and $u_0 \in V$.

We have the following theorem of existence.

Theorem 2.1. *Assume that assumptions H(A),H(B),H(φ),H(J), H($\mathcal{C}$) H(f) and H(L) holds. Then the hyberbolic variational inequality (1) has a solution.*

The proof of Theorem 2.1 is based on three basic steps.

1. We reformulate the hyperbolic variational inequality (1) as an inclusion.
2. We define time discrete family problems corresponding to the inclusion which solved by surjectivity theorem.
3. We prove a convergence result. Hence, we deduce that the hyperbolic variational inequality (1) has a solution.

3 A Dynamic contact frictional problem for viscoelastic materials

In this section, we consider a dynamic contact frictional problem for viscoelastic materials and we prove existence of weak solution by using the abstract result in Section 2. The friction condition is described with the evolutionary version of Coulomb law of dry friction, More details can be found in [2].

4 Conclusion

As conclusion, it is evident that this study has shown that the hyperbolic variational inequality (1) has a solution. Further study of the issue would be of interest when the viscosity term is vanished in order to obtain an existence result for an elastodynamic Signorini problem with Coulomb friction law.

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JUSTIFICATION OF THE TWO-DIMENSIONAL EQUATIONS OF VON KÁRMÁN SHELLS

MARWA LEGOUGUI AND ABDERREZAK GHEZAL

Abstract. In this work, using the method of asymptotic expansions with the thickness as the "small" parameter, we show that the three-dimensional for a nonlinearly elastic shells of Saint Venant-Kirchhoff material with boundary conditions of von Kármán's type, written in curvilinear coordinates reduces to two-dimensional von Kármán model.

2010 Mathematics Subject Classification. 74B20, 74K25, 74G10.

Keywords and phrases. Nonlinear elasticity, shell theory, von Kármán conditions, asymptotic analysis.

1. Introduction

The von Kármán equations are two-dimensional model for a nonlinearly elastic plate subjected to boundary conditions of von Kármáns type. They were initially proposed by von Kármán [7], which originating from continuum mechanics and play an important role in applied mathematics. Next, these equations are extended to Marguerre- von Kármán equations for a nonlinearly elastic shallow shell by Marguerre [6]. Then Ciarlet [1] and Ciarlet and Paumier [2] justified the both of previous models by formal asymptotic methods.

The asymptotic methods can be used for justifying the two-dimensional models of elastic plates and shells starting from the three-dimensional models. Numerous works have been devoted to plates and shells in static case (see, e.g., [3]-[4]). For dynamical case, we refer to Ghezal and Chacha [5].

A natural question arises as: How to extend the von Kármán and Marguerre-von Kármán equations to the more general geometry of a shell?

2. Three-dimensional problem

Throughout this paper, we use the following conventions and notations: Greek indices (except for ε), belong to the set $\{1, 2\}$, while Latin indices belong to the set $\{1, 2, 3\}$, the symbols of differentiation $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_i^\varepsilon = \frac{\partial}{\partial x_i^\varepsilon}$, $\hat{\partial}_i^\varepsilon = \frac{\partial}{\partial \hat{x}_i^\varepsilon}$, δ_{ij} the Kronecker symbols. The summation convention with respect to repeated indices is systematically used.

Consider a nonlinearly elastic shell with middle surface $S = \theta(\bar{\omega})$ and thickness $2\varepsilon > 0$, its constituting material is a Saint Venant-Kirchhoff material with Lam constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$, where ω is a domain in $\mathbb{R}^2$ with a boundary γ, and $\theta : \bar{\omega} \longrightarrow E^3$ is a smooth enough injective immersion, such that the two vectors $a_\alpha(y) = \partial_\alpha \theta(y)$ are linearly independent at all points $y \in \bar{\omega}$, which form the covariant basis of the tangent plane to the surface $S = \theta(\bar{\omega})$.

We define the mapping $\Theta : \bar{\Omega}^\varepsilon \longrightarrow \mathbb{R}^3$ as follow:

$$\Theta(x^\varepsilon) = \theta(y) + x_3^\varepsilon a_3(y),\ \forall (y, x_3) \in \bar{\Omega}^\varepsilon,$$

where

$$a_3(y) = a^3(y) = \frac{a_1 \wedge a_2}{|a_1 \wedge a_2|}.$$

The mapping Θ is assumed to be an immersion, the three vectors $g_i(x) = \partial_i \Theta(x)$, which are linearly independent at all points $x \in \bar{\Omega}$, thus form the covariant basis at $\hat{x} = \Theta(x) \in \bar{\hat{\Omega}}$. For each $\varepsilon > 0$, we define the sets:

$$\bar{\Omega}^\varepsilon = \bar{\omega} \times [-\varepsilon, \varepsilon],\ \Gamma_0^\varepsilon = \gamma \times [-\varepsilon, \varepsilon],\ \Gamma_\pm^\varepsilon = \omega \times \{\pm\varepsilon\}.$$

The shell is subjected to applied body forces in its interior $\hat{\Omega}^\varepsilon = \Theta(\Omega^\varepsilon)$, of density $(\hat{f}_i^\varepsilon) : \hat{\Omega}^\varepsilon \longrightarrow \mathbb{R}^3$, to applied surface forces on the upper and the lower faces $\hat{\Gamma}_\pm^\varepsilon = \Theta(\Gamma_\pm^\varepsilon)$, of density $(\hat{l}_i^\varepsilon) : \hat{\Gamma}_+^\varepsilon \cup \hat{\Gamma}^\varepsilon \longrightarrow \mathbb{R}^3$, and to horizontal forces on the lateral face $\hat{\Gamma}_0^\varepsilon = \Theta(\Gamma_0^\varepsilon)$, which we are given the averaged density $(\hat{h}_1^\varepsilon, \hat{h}_2^\varepsilon, 0) : \Theta(\gamma) \longrightarrow \mathbb{R}^3$, after integration across the thickness of the shell. The displacement verifies specific conditions on the lateral face, in that only horizontal displacements are allowed along every vertical segment of the lateral face.

We define the espace

$$\mathbf{V}(\hat{\Omega}^\varepsilon) = \{\hat{v}^\varepsilon = (\hat{v}_i^\varepsilon) \in W^{1,4}(\hat{\Omega}^\varepsilon; \mathbb{R}^3);\ \hat{v}_i^\varepsilon = 0 \text{ on } \hat{\Gamma}_0^\varepsilon,$$
$$\hat{v}_\alpha^\varepsilon \text{ is independent of } \hat{x}_3^\varepsilon \text{ and } \hat{v}_3^\varepsilon = 0 \text{ on } \hat{\Gamma}_1^\varepsilon\},$$
$$\hat{\Sigma}^\varepsilon = \{\hat{\tau}^\varepsilon = (\hat{\tau}_{ij}^\varepsilon) \in (L^2(\hat{\Omega}^\varepsilon))^9; \hat{\tau}_{ij}^\varepsilon = \hat{\tau}_{ji}^\varepsilon\}.$$

The unknown displacement field $\hat{u}^\varepsilon = (\hat{u}_i^\varepsilon) : \{\hat{\Omega}^\varepsilon\}^- \to \mathbb{R}^3$ satisfy the following three-dimensional von Kármán shell problem in cartesian coordinates:

$$(C.\hat{P}^\varepsilon) \begin{cases} -\hat{\partial}_j^\varepsilon(\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon) = \hat{f}_i^\varepsilon \text{ in } \hat{\Omega}^\varepsilon, \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon = \hat{l}_i^\varepsilon \text{ on } \hat{\Gamma}_-^\varepsilon \cup \hat{\Gamma}_+^\varepsilon, \\ \hat{u}_i^\varepsilon = 0 \text{ on } \hat{\Gamma}_0^\varepsilon, \\ \begin{cases} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{+\varepsilon} (\hat{\sigma}_{\alpha\beta}^\varepsilon + \hat{\sigma}_{k\beta}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_\alpha^\varepsilon)\nu_\beta dx_3^\varepsilon = \hat{h}_\alpha^\varepsilon \text{ on } \theta(\gamma_1), \\ \hat{u}_\alpha^\varepsilon \text{ independent of } \hat{x}_3^\varepsilon \text{ on } \hat{\Gamma}_1^\varepsilon, \\ \hat{u}_3^\varepsilon = 0 \text{ on } \hat{\Gamma}_1^\varepsilon, \end{cases} \end{cases}$$

the Piola-Kirchhoff stress tensor $(\hat{\sigma}_{ij}^\varepsilon)$ and the Green-Saint Venant strain tensor $(\hat{E}_{ij}(\hat{u}^\varepsilon))$ are given by

$$\begin{cases} \hat{\sigma}_{ij}^\varepsilon = \lambda^\varepsilon \hat{E}_{pp}^\varepsilon(\hat{u}^\varepsilon)\delta_{ij} + 2\mu^\varepsilon \hat{E}_{ij}^\varepsilon(\hat{u}^\varepsilon), \\ \hat{E}_{ij}^\varepsilon(\hat{u}^\varepsilon) = \frac{1}{2}(\hat{\partial}_i^\varepsilon \hat{u}_j^\varepsilon + \hat{\partial}_j^\varepsilon \hat{u}_i^\varepsilon + \hat{\partial}_i^\varepsilon \hat{u}_m^\varepsilon \hat{\partial}_j^\varepsilon \hat{u}_m^\varepsilon), \end{cases}$$

3. Two-dimensional models

We define the covariant components u_m^ε of the displacement field by:

$$\hat{u}_i^\varepsilon(\hat{x}^\varepsilon)\hat{e}^i = u_m^\varepsilon(x^\varepsilon) g^{m,\varepsilon}(x^\varepsilon),\ \forall \hat{x}^\varepsilon = \Theta(x^\varepsilon) \in \{\bar{\hat{\Omega}}^\varepsilon\},$$

where $(\hat{e}^i)$ denotes the canonical basis of $\mathbb{R}^3$ and $(g^{m,\varepsilon}(x^\varepsilon))$ is the contravariant basis at the point $\hat{x^\varepsilon}$. The covariant basis $(g_i^\varepsilon(x^\varepsilon))$, is given by

$$g_i^\varepsilon(x^\varepsilon) = \partial_i^\varepsilon \Theta(x^\varepsilon),\ \forall x^\varepsilon \in \Omega^\varepsilon.$$

The Christoffel symbols and the covariant and contravariant components of the metric tensor, defined by

$$\Gamma_{ij}^{k,\varepsilon} = \partial_i^\varepsilon g_j^\varepsilon . g^{k,\varepsilon}, \; g_{ij}^\varepsilon = g_i^\varepsilon . g_j^\varepsilon \; g^{ij,\varepsilon} = g^{i,\varepsilon}.g^{j,\varepsilon}.$$

We define the contravariant components of the applied forces by

$$\hat{f}_i^\varepsilon(\hat{x}^\varepsilon)\hat{e}^i = f^{i,\varepsilon}(x^\varepsilon) g_i^\varepsilon(x^\varepsilon), \; \forall x^\varepsilon \in \Omega^\varepsilon,$$
$$\hat{l}_i^\varepsilon(\hat{x}^\varepsilon)\hat{e}^i = l^{i,\varepsilon}(x^\varepsilon) g_i^\varepsilon(x^\varepsilon), \; \forall x^\varepsilon \in \Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon,$$
$$\bar{h}_i^\varepsilon(y) e^i = h^{i,\varepsilon}(y) g_i^\varepsilon(y, x_3), \forall y \in \gamma_1.$$

We define the following space

$$\mathbf{V}(\Omega) = \{v = (v_i) \in W^{1,4}(\Omega; \mathbb{R}^3); v = 0 \text{ on } \Gamma_0,$$
$$v_\alpha \text{ is independent of } x_3, v_3 = 0 \text{ on } \Gamma_1\}.$$

Assume that the scaled unknown $u(\varepsilon) = u_i^\varepsilon$ admits a formal asymptotic expansion of the form

$$u(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + ...,$$

with

$$u^0 \in \mathbf{V}(\Omega) \quad \textit{and} \quad u^p \in W^{1,4}(\Omega), \; \forall p \geq 1.$$

The components of the applied forces are of the form

$$f^{i,\varepsilon}(x^\varepsilon) = f^{i,0}(x),$$
$$l^{i,\varepsilon}(x^\varepsilon) = \varepsilon l^{i,1}(x),$$
$$h^{\alpha,\varepsilon}(y) = h^{\alpha,0}(y),$$

where the functions $f^{i,0} \in L^2(\Omega)$ and $l^{i,1} \in L^2(\Gamma_+ \cup \Gamma_-)$ and $h^{\alpha,0} \in L^2(\gamma)$ are independent of ε.

We now give the main results of this work

Theorem 3.1. *The leading term u^0 is independent of the transverse variable x_3 and it can be identified with ζ^0, which satisfies the following two-dimensional variational problem:*

$$\zeta^0 \in \mathbf{V}(\omega) = \{\eta \in W^{1,4}(\omega); \eta = 0 \; \textit{on} \; \gamma_0, \eta_3 = 0 \; \textit{on} \; \gamma_1\},$$

$$\int_\omega a^{\alpha\beta\sigma\tau} E^0_{\sigma\|\tau} F^0_{\alpha\|\beta}(\eta)\sqrt{a}dy = \int_\omega P^{i,0}\eta_i\sqrt{a}dy + 2\int_{\gamma_1} h^{\beta,0}\eta_\alpha d\gamma,$$

for all $\eta = (\eta_i) \in \mathbf{V}(\omega)$, where

$$E^0_{\alpha\|\beta} = \frac{1}{2}(\zeta^0_{\alpha\|\beta} + \zeta^0_{\beta\|\alpha} + a^{mn}\zeta^0_{m\|\alpha}\zeta^0_{n\|\beta}),$$
$$F^0_{\alpha\|\beta}(\eta) = \frac{1}{2}(\eta_{\alpha\|\beta} + \eta_{\beta\|\alpha} + a^{mn}\{\zeta^0_{m\|\alpha}\eta_{n\|\beta} + \zeta^0_{n\|\beta}\eta_{m\|\alpha}\}),$$
$$\eta_{\alpha\|\beta} = \partial_\beta\eta_\alpha - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3 \; \textit{and} \; \eta_{3\|\beta} = \partial_\beta\eta_3 + b^\sigma_\beta\eta_\sigma,$$
$$a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}),$$
$$P^{i,0} = \int_{-1}^1 f^{i,0} + l^{i,1}_- + l^{i,1}_+ \quad \textit{and} \quad l^{i,1}_\pm = l^{i,1}(., \pm 1).$$

4. Conclusion

An application of the technics from formal asymptotic analysis to the three-dimensional model of nonlinearly elastic shells with boundary conditions von Kármán type, made of a Saint Venant-Kirchhoff material, shows that the leading term of the expansion is characterized by a two-dimensional model.

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L^∞-ASYMPTOTIC BEHAVIOR OF A FINITE ELEMENT METHOD FOR A SYSTEM OF PARABOLIC QUASI-VARIATIONAL INEQUALITIES WITH NONLINEAR SOURCE TERMS

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1

Abstract. This paper is an extension and a generalization of the previous results, cf. [3,6,8,11]. It is devoted to studying the finite element approximation of the non coercive system of parabolic quasi-variational inequalities related to the management of energy production problem. Specifically, we prove optimal L1-asymptotic behavior of the system of evolutionary quasi-variational inequalities with nonlinear source terms using the finite element spatial approximation and the subsolutions method.

Key Words : Quasi-variational inequalities, asymptotic behavior, subsolutions method, finite elements approximation, L^∞-error estimate.

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LIPSCHITZ GLOBAL OPTIMIZATION PROBLEM AND α-DENSE CURVES

DJAOUIDA GUETTAL AND MOHAMED RAHAL

Abstract. In this paper, we study a coupling of the Alienor method with the algorithm of Piyavskii-Shubert. The classical multidimensional global optimization methods involves great difficulties for their implementation to high dimensions. The Alienor method allows to transform a multivariable function into a function of a single variable for which it is possible to use efficient and rapid method for calculating the the global optimum. This simplification is based on the using of a reducing transformation called Alienor.

Keywords. The Alienor method, Algorithm of Piyavskii-Shubert, Global optimization method, α-dense curves

1. Define the problem

Let us consider the following lipschitz global optimization problem

$$\begin{cases} \min F(x) \\ \text{subject to } g_i(x) \leq 0,\, i \in I \\ x \in \Omega \end{cases}$$

where $x = (x_1, \cdots, x_n)^T$ is the real vector of $\mathbb{R}^n$ represents the n variables, I is a finite index set and Ω is a compact in $\mathbb{R}^n$.

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MAPPED LEGENDRE SPECTRAL METHODS FOR SOLVING A QUADRATIC HAMMERSTIEN INTEGRAL EQUATION ON THE HALF LINE

RADJAI ABIR AND RAHMOUNE AZEDINE

Abstract. In this work, we introduce a new extension of the Legendre spectral collocation method has been proposed for the numerical solution of a quadratic Hammerstien integral equation on the half-line. The main idea is to map the infinite interval to a finite one and use Legendre spectral-collocation method to solve the mapped integral equation in the finite interval. Numerical examples are presented to illustrate the accuracy of the method.

Keywords and phrases. A quadratic Hammerstien integral equation, Half-line, Mapped Legendre, Lagrange interpolation, Collocation points, Error estimate.

1. Define the problem

The main objective of my work is to extend the Legendre spectral method to a quadratic Hammerstien integral equation on the half-line of the forme:

$$u(x) = a(x) + f(x, u(x)) \int_0^\infty k(x,t)g(t,u(t))dt \quad x \in \mathbb{R}^+ \tag{1}$$

Where $k(x,t), g(t,u(t)), a(x), and\ f$ are given continuous functions and $u(x)$ is unknown function.
In [1] Jozef Banas and Donal O'reganand and others using the technique of measures of noncompactness with the classical Schauder fixed point principle. Such an approach permits us to obtain our existence results under rather general assumptions and In [2] the same others applying the Darbo fixed point theorem to prove that the equation (1) has solution in the class of real funtions defined bounded continous on the real half axis and having limits at infinity. The present paper focuses on the numerical solution of this kind of equations.

The method of solution is based on the reduction of the problem to a finite interval $[-1, 1]$ by means of a suitable family of mappings so that the resulting singular equation can be accurately solved using spectral collocation at the Legendre-Gauss points. Several selected numerical examples are presented and discussed to illustrate the application and effectiveness of the proposed approach.

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Abstract :

In this article, we study the transmission of COVID-19 in the human population, notably between potential people and infected people of all age groups. Our objective is to reduce the number of infected people, in addition to increasing the number of individuals who recovered from the virus and are protected. We propose a mathematical model with control strategies using two variables of controls that represent respectively, the treatment of patients infected with COVID-19 by subjecting them to quarantine within hospitals and special places and using masks to cover the sensitive body parts. Pontryagin's Maximum principle is used to characterize the optimal controls and the optimality system is solved by an iterative method. Finally, numerical simulations are presented with controls and without controls. Our results indicate that the implementation of the strategy that combines all the control variables adopted by theWorld Health Organization (WHO), produces excellent results similar to those achieved on the ground in Morocco.

Keywords:

COVID-19. Mathematical Model. Age groups. Pontryagin's Maximum Principle. Optimal Control.

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MATHEMATICAL ANALYSIS OF A DYNAMIC PIEZOELECTRIC CONTACT PROBLEM WITH FRICTION

KHEZZANI RIMI

Abstract. ...

2010 Mathematics Subject Classification. xxxx, xxxx, xxxx.

Keywords and phrases. elastic-viscoplastic piezoelectric materials; internal state variable; normal compliance; wear; evolution equations; fixed point.

1. Problem Statement

We consider the following physical setting. Let us consider two electro-elastic- viscoplastics bodies, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d (d = 2, 3)$. For each domain Ω^κ, the boundary Γ^κ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^κ, Γ_2^κ and Γ_3^α on one hand, and on two measurable parts Γ_a^α and Γ_b^α, on the other hand, such that $meas(\Gamma_1^\alpha) > 0$, $meas(\Gamma_a^\alpha) > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The Ω^α body is submitted to $\boldsymbol{f}_0^\alpha$ forces and volume electric charges of density q_0^α. The bodies are assumed to be clamped on $\Gamma_1^\alpha \times [0, T]$. The surface tractions $\boldsymbol{f}_2^\alpha$ act on $\Gamma_2^\alpha \times [0, T]$. We also assume that the electrical potential vanishes on $\Gamma_a^\alpha \times [0, T]$ and a surface electric charge of density q_2^α is prescribed on $\Gamma_b^\alpha \times [0, T]$. The two bodies can enter in bilateral contact with friction along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. The bodies are in contact with friction and wear, over the contact surface Γ_3. we introduce the wear function $\omega : \Gamma_3 \times [0, T] \longrightarrow \mathbb{R}^+$ which measures the wear of the surface. The wear is identified as the normal depth of the material that is lost. Let g be the intial gap between the two bodies. Let p_ν and p_τ denote the normal and tangential compliance functions. We denote by $\boldsymbol{v}^*$ and $\alpha^* = \|\boldsymbol{v}^*\|$ the tangential velocity and the tangential speed at the contact surface between the two bodies. We use the modified version of Archard's law:

$$\dot{\omega} = -\lambda_0 \boldsymbol{v}^* \sigma_\nu.$$

To describe the evolution of wear, where $\lambda_0 > 0$ is a wear coefficient. We introduce the unitary vector $\delta : \Gamma_3 \longrightarrow \mathbb{R}^d$ defined by $\delta = \boldsymbol{v}^*/\|\boldsymbol{v}^*\|$. When the contact arises, some material of the contact surfaces worn out and immediately removed from the system. This process is measured by the wear function ω. With these assumptions above, the classical formulation of the mechanical frictional contact problem with wear between two electro-elastic-viscoplastics bodies is the following.

Problem P. For $\alpha = 1, 2$, find a displacement field $\boldsymbol{u}^\alpha : \Omega^\alpha \times [0, T] \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\alpha : \Omega^\alpha \times [0,T] \longrightarrow \mathbb{S}^d$, an electric potential field $\psi^\alpha : \Omega^\alpha \times [0,T] \longrightarrow \mathbb{R}$, a wear $\omega : \Gamma_3 \times [0,T] \longrightarrow \mathbb{R}^+$ and a electric displacement field $\boldsymbol{D}^\alpha : \Omega^\alpha \times [0,T] \longrightarrow \mathbb{R}^d$ and an internal state variable field $\beta^\alpha : \Omega^\alpha \times [0,T] \longrightarrow \mathbb{R}^m$ such that

(1)
$$\boldsymbol{\sigma}^\alpha(t) = \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(t)) + \mathcal{G}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(t)) + (\mathcal{E}^\alpha)^* \nabla \psi^\alpha(t) + \int_0^t \mathcal{F}^\alpha\Big(\boldsymbol{\sigma}^\alpha(s) - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(s)) - (\mathcal{E}^\alpha)^* \nabla \psi^\alpha(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(s)), \beta^\alpha(s)\Big)\, ds \quad \text{in } \Omega^\alpha \times [0,T],$$

(2)
$$\dot{\beta}^\alpha(t) = \Theta^\alpha\big(\boldsymbol{\sigma}^\alpha(t) - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(t)) - (\mathcal{E}^\alpha)^* \nabla \psi^\alpha(t), \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(t)), \beta^\alpha(t)\big) \quad \text{in } \Omega^\alpha \times [0,T],$$

$$\boldsymbol{D}^\alpha(t) = \mathcal{E}^\alpha \varepsilon(\boldsymbol{u}^\alpha(t)) - \mathcal{B}^\alpha \nabla \psi^\alpha(t) \quad \text{in } \Omega^\alpha \times [0,T], \tag{3}$$

$$\rho^\alpha \ddot{\boldsymbol{u}}^\alpha = \mathrm{Div}\, \boldsymbol{\sigma}^\alpha + \boldsymbol{f}_0^\alpha \quad \text{in } \Omega^\alpha \times [0,T], \tag{4}$$

$$\mathrm{div}\, \boldsymbol{D}^\alpha - q_0^\alpha = 0 \quad \text{in } \Omega^\alpha \times [0,T], \tag{5}$$

$$\boldsymbol{u}^\alpha(t) = 0 \quad \text{on } \Gamma_1^\alpha \times [0,T], \tag{6}$$

$$\boldsymbol{\sigma}^\alpha \boldsymbol{\nu}^\alpha = \boldsymbol{f}_2^\alpha \quad \text{on } \Gamma_2^\alpha \times [0,T], \tag{7}$$

$$\sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \quad \text{where} \quad \sigma_\nu = -p_\nu(u_\nu - \omega - g) \quad \text{on } \Gamma_3 \times [0,T], \tag{8}$$

(9)
$$\boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \quad \text{where} \quad \boldsymbol{\sigma}_\tau = -p_\tau(\boldsymbol{u}_\nu - \omega - g)\frac{\boldsymbol{v}^*}{\|\, \boldsymbol{v}^* \,\|} \quad \text{on } \Gamma_3 \times [0,T],$$

$$u_\nu^1 + u_\nu^2 = 0 \quad \text{on } \Gamma_3 \times [0,T], \tag{10}$$

$$\dot{\omega} = -\lambda_0 \alpha^* \sigma_\nu \quad \text{on } \Gamma_3 \times [0,T], \tag{11}$$

$$\psi^\alpha(t) = 0 \quad \text{on } \Gamma_a^\alpha \times [0,T], \tag{12}$$

$$\boldsymbol{D}^\alpha . \boldsymbol{\nu}^\alpha = q_2^\alpha \quad \text{on } \Gamma_b^\alpha \times [0,T], \tag{13}$$

$$\boldsymbol{u}^\alpha(0) = \boldsymbol{u}_0^\alpha, \quad \dot{\boldsymbol{u}}^\alpha(0) = \boldsymbol{v}_0^\alpha, \quad \beta^\alpha(0) = \beta_0^\alpha \quad \text{in } \Omega^\alpha, \tag{14}$$

$$\omega(0) = \omega_0 \quad \text{on } \Gamma_3. \tag{15}$$

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MATHEMATICAL STUDY AIMING AT ADOPTING AN EFFECTIVE STRATEGY TO COEXIST WITH CORONAVIRUS PANDEMIC

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ABSTRACT. In this paper, we propose a discrete mathematical model that describes the evolution of the "covid-19" virus in a human population and the efforts made to control it. Our objective is to develop a simple, logical and an optimal strategy to reduce the negative impact of this infectious disease on countries. This objective is achieved through maximizing the number of people applying the preventive measures recommended by WHO against the pandemic in order to reduce the infection as much as possible. The tools of optimal control theory were used in this study, in particular Pontryagin's maximum principle.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 93A30, 49J15.

KEYWORDS AND PHRASES. optimal control; SARS-COV-2; mathematical model; discrete model; COVID-19.

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MÉTHODE DE HALLEY DANS UN ESPACE ULTRAMÉTRIQUE

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Abstract. Ce travail est une application intéressante des outils de l'analyse numérique à la théorie des nombres p-adiques avec p un nombre premier. On verra comment utiliser la méthode numérique élémentaire de Halley pour calculer les premiers chiffres des développements finis p-adiques des racines cubiques $\sqrt[3]{a}$ d'un nombre p-adique $a \in \mathbb{Q}_p$ à l'aide d'une suite $(x_n)_n$ de nombres p-adiques construite par la méthode de Halley. La vitesse de sa convergence et le nombre d'itérations nécessaires pour que $(x_n)_n$ soit proche de $\sqrt[3]{a}$ avec une précision donnée M qui représente le nombre de chiffres p-adiques dans le développement de $\sqrt[3]{a}$ sont calculés.

2010 Mathematics Subject Classification. 26E30, 11E95, 34K28.

Keywords and phrases. valuation p-adique, norme p-adique, nombre p-adique, racine cubique, développement p-adique, Méthode de Halley, vitesse de convergence.

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NEW RESULTS ON THE CONFORMABLE FRACTIONAL ELZAKI TRANSFORM

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Abstract. The fractional calculus has been used in the pure and applied branches of science and engineering in the present centuries. Resently several types of fractional definitions are given, such as Riemann-Liouville, Grunwald-Letnikov , Caputo's fractional definition and a simple definition called " Conformable fractional derivative" was proposed by Khalil and al.(2014). The definition of conformable fractional derivative is similar to the limit based definition of known derivative. This derivative obeys both rule which other popular derivatives do not satisfy such as the derivative product of two functions, Gronwall's inequality, Taylor power series expansions, chain rule, etc In this work we introduce a new results of Elzaki transform with a conformable fractional motivated by the fractional Laplace transform.

2010 Mathematics Subject Classification. 26A33, 42B10, 45F15.

Keywords and phrases. : Conformable fractional derivative, Laplace transform, Elzaki transform.

1. Define the problem

The objective of this work is to study the following problem: how to calculate the Elzaki transformation of a conformable fractional derivative ?. Our work is divided into two parts the first part concerns, some reminders of the conformable fractional derivative and of the Elzaki transform, the second part, will be devoted to generalise the formula of Elzaki transform to the conformable fractional order and some interesting rules of this transform and conformable fractional Laplace transform.

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NUMERICAL SOLUTION FOR FRACTIONAL ODES VIA REPRODUCING KERNEL HILBERT SPACE METHOD: APPLICATION TO A BIOLOGICAL SYSTEM

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ABSTRACT. In this study, the numerical solutions for an essential fractional ordinary differential equation has been investigated with the aid of the reproducing kernel Hilbert space method (RKHSM). The convergence analysis associated with the RKHSM is studied to provide the theoretical basis of the suggested approach for solving the considered problem. The numerical simulations are presented to show the accuracy and reliability of the proposed method.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 46E22, 35R11.

KEYWORDS AND PHRASES. Fractional ordinary differential equations, Reproducing kernel Hilbert space method, Caputo fractional derivative, Convergence analysis, Approximate solution.

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NUMERICAL SOLUTION OF A CLASS OF WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS BY USING AN ITERATIVE COLLOCATION METHOD

KHEDIDJA KHERCHOUCHE AND AZZEDDINE BELLOUR

Abstract. An iterative collocation method based on the use of Lagrange polynomials is developed for the numerical solution of a class of nonlinear weakly singular Volterra integral equations. The approximate solution is given by explicit formulas and there is no algebraic system needed to be solved. The error analysis of the proposed numerical method is studied theoretically. Some numerical examples are given to show the validity of the presented method.

2000 Mathematics Subject Classification 45D05, 65R20

Keywords and phrases. nonlinear weakly singular Volterra integral equation, Collocation method; Iterative Method, Lagrange polynomials.

1. Define the problem

In this work,we consider the following nonlinear weakly singular Volterra integral equations,

$$x(t) = f(t) + \int_0^t p(t,s)k(t,s,x(s))ds, t \in I = [0,T], \tag{1}$$

where the functions f, k are sufficiently smooth and $p(t,s) = \frac{s^{\mu-1}}{t^{\beta}}$,$\beta > 0$, $\mu \geq \beta + 1$.

Equations with this kind of kernel have a weak singularity at $t = 0$.

Equation (1) is a particular case of the nonlinear of the so-called "cordial" integral equations, which were introduced by Vainikko in [1].

The existence and uniqueness result in $C^m([0,T])$ for the nonlinear cordial equations was obtained in [2].

Cordial integral equations are frequently encountered in some heat conduction problems with mixed-type boundary conditions [4, 5, 3].

The main goal of this work is to develop a collocation method based on the use of Lagrange polynomials for the numerical solution of this equation.

The main advantages of this method that, is easy to implement, has high order of convergence and the coefficients of approximate solution are determined by using iterative formulas without solving any system of algebraic equations.

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NUMERICAL SOLUTION OF LINEAR FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. There are several phenomenons and problems in physics, biology and many other fields which are modelled by the integro-differential equations. Recently, various researchers have constructed different methods to find an approximation solution for these types of equations. The propose of our work is to study the solution's existence and uniqueness for the linear integro-differential Fredholm equation then we construct an approximate solution by using the Nyström method. The study is based on: Firstly, we transform the linear integro-differential Fredholm equation to a linear Fredholm integral system and we build a sufficient condition to show the solution's existence and uniqueness of the system. Secondly, we apply Nyström method, which discretizes the system of integro-differential equations into solving a linear algebric system. Finally, we give a theorem to prove the convergence of the approximate solution to the exact solution in $C^1[a,b]$.

2010 Mathematics Subject Classification. 45B05, 45J05, 47G20, 34K28, 45L05, 65R20.

Keywords and phrases. Fredholm integral equation, system of integral equations, integro-differential equations, Nyström method.

1. Problem position

Let $X = C^1[a,b]$ be the Banach space with the norm:

$$||u||_X = \max_{a\le x\le b}|u(x)| + \max_{a\le x\le b}|u'(x)|$$

Let $u \in X$ be the solution of the following linear Fredholm integro-differential equation:

$$(1)\quad \forall x\in[a,b],\ \lambda u(x) = \int_a^b K_1(x,t)u(t)\,dt + \int_a^b K_2(x,t)u'(t)\,dt + f(x),$$

where, λ is a complexe parameter, f and K_i, for $i=1,2$ are given functions. We suppose that K_i, for $i=1,2$ satisfied the following hypotheses:

$$(H_1)\,\Big\|\ \frac{\partial K_i}{\partial x}(x,t) \in C^0([a,b]^2,\mathbb{R}).$$

Then, the derivative u' is given implicitly by

$$(2)\qquad \forall x\in[a,b],\ \lambda u'(x) = \int_a^b \frac{\partial K_1}{\partial x}(x,t)u(t)\,dt + \int_a^b \frac{\partial K_2}{\partial x}(x,t)u'(t)\,dt + f'(x).$$

Theorem 1.1. *If* $|\lambda| > (b-a)\left(\max_{a\le x,t\le b}|K_i(x,t)| + \max_{a\le x,t\le b}\frac{\partial K_i}{\partial x}(x,t)\right)$, *for* $i=1,2$ *the system* (**??**)-(**??**) *has a unique solution in* X.

1

Now, we construct an approximation of the system (**??**)-(**??**) based on the Nyström method. First, we define

$$\Delta_n = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\},$$

be an uniforme subdivision of interval [a,b] with $x_j = a + jh$ and $h = \frac{b-a}{n}$, $\forall n \geq 1$.
Applying the Nyström method (see [**?**]) to our equation.
We obtain, $\forall x \in [a, b]$

$$\begin{cases} \lambda u_n(x) & = h\sum_{j=0}^{n}\omega_j K_1(x_i, x_j)u_n(x_j) + h\sum_{j=0}^{n}\omega_j K_2(x_i, x_j)u_n'(x_j) + f(x), \\ \lambda u_n'(x) & = h\sum_{j=0}^{n}\omega_j \frac{\partial K_1}{\partial x}(x_i, x_j)u_n(x_j) + h\sum_{j=0}^{n}\omega_j \frac{\partial K_2}{\partial x} K_2(x_i, t_j)u_n'(x_j) + f'(x), \end{cases}$$

where, $\{\omega_i\}_{0\leq i\leq n}$ called quadrature weights, such that $\sup_{n\geq 1}\sum_{i=0}^{n}|\omega_i| < \infty$.

Theorem 1.2. *Under the assumption H_1, the approximation solution u_n converges to the exact solution in X.*

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NUMERICAL SOLUTION OF SECOND ORDER LINEAR DELAY DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS BY USING TAYLOR COLLOCATION METHOD

AZZEDDINE BELLOUR AND HAFIDA LAIB

Abstract. The main purpose of this work is to provide a numerical approach for linear second order differential and integro-differential equations with constant delay. An algorithm based on the use of Taylor polynomials is developed to construct a collocation solution $u \in S_m^{(1)}(\Pi_N)$ for approximating the solution of second order linear DDEs and DIDEs. It is shown that this algorithm is convergent. Some numerical examples are included to demonstrate the validity of the presented algorithm.

2010 Mathematics Subject Classification. 45L05, 65R20.

Keywords and phrases. Second order delay linear differential equations, integro-differential equations, Collocation method, Taylor polynomials.

1. Define the problem

In this work, we consider the second order linear Volterra integro-differential equations (VIDEs) with constant delay $\tau > 0$ of the form:

$$\begin{aligned} x''(t) &= g(t) + L_0(t)x(t) + L_1(t)x'(t) + M_0(t)x(t-\tau) + M_1(t)x'(t-\tau) \\ &\quad + \int_0^t k_1(t,s)x(s)ds + \int_0^{t-\tau} k_2(t,s)x(s)ds, \end{aligned} \tag{1}$$

for $t \in [0,T]$ and $x(t) = \Phi(t)$ for $t \in [-\tau, 0]$. In the following we assume that the given functions $g, k_1, k_2, L_0, L_1, M_0, M_1$ and Φ are sufficiently smooth. Furthermore, we suppose that

$$\begin{aligned} \Phi''(0) &= g(0) + L_0(0)\Phi(0) + L_1(0)\Phi'(0) + M_0(0)\Phi(-\tau) + M_1(0)\Phi'(-\tau) \\ &\quad - \int_{-\tau}^0 k_2(0,s)\Phi(s)ds \end{aligned}$$

Existence and uniqueness of the solution for Equation (1) can be easily proved by using the conjunction of the iterative technique with Banach's fixed point theorem on the intervals $[\tau, 2\tau], [2\tau, 3\tau], \ldots,$.
The main goal of this work is to develop a collocation method based on the use of Taylor polynomials for the numerical solution of the second order linear VIDEs (1) with constant delay. The advantage of this collocation method is: This method is explicit and direct, has a convergence order, and there is no algebraic system needed to be solved, which makes the proposed algorithm very effective, easy to implement.

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ON THE EXISTENCE OF A SOLUTION OF A NONLINEAR EVOLUTION DAM PROBLEM

MESSAOUDA BEN ATTIA AND ELMEHDI ZAOUCHE

Abstract. Consider an arbitrary heterogeneous porous medium Ω of $\mathbb{R}^2$ with an impermeable horizontal bottom and suppose that the function corresponding to the Darcy's law is coercive only on one direction. We adapt the Poincaré inequality for Ω and we apply techniques as in [1] to prove the existence of a solution to a nonlinear evolution dam problem.

2010 Mathematics Subject Classification. 35A02, 35B35, 76S05.

Keywords and phrases. Nonlinear evolution dam problem; heterogeneous porous medium with an impermeable horizontal bottom; existence.

1. The problem

Let A, B, D and T be real numbers such that $B > A$, $T > 0$ and let Ω be a bounded domain in $\mathbb{R}^2$ with locally Lipschitz boundary $\partial\Omega := \Gamma$ and horizontal bottom $\Gamma_1 = [A, B] \times \{D\}$. Ω represents a porous medium. The boundary Γ is divided into two parts: an impervious part Γ_1 and a pervious Γ_2 which is a nonempty relatively open subset of Γ. We are interested in the motion of an incompressible fluid in Ω and in a time interval $[0, T]$ and we are looking for a pair $(\mathbf{p}, \chi)$ where $\mathbf{p}$ is the pressure of the fluid and χ a function characterizing the wet part W of the dam. Let φ be a nonnegative Lipschitz continuous function defined in $\overline{\Omega \times (0, T)} := \overline{Q}$ which represents the assigned pressure on $\Gamma_2 \times (0, T) = \Sigma_2$. The velocity v and the pressure of the fluid in W are related by a nonlinear Darcy's law:

$$v = -a(x, (\mathbf{p} + x_2)_{x_2}),$$

where $x = (x_1, x_2)$ and $a : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying $x \mapsto a(x, r)$ is measurable for all $r \in \mathbb{R}$, the function $r \mapsto a(x, r)$ is continuous for a.e. $x \in \Omega$ and for some constants $p > 1$ and $\lambda, \Lambda > 0$:

$$\forall r \in \mathbb{R},\ \text{a.e. } x \in \Omega : \quad \lambda |r|^p \le a(x, r) r,$$
$$\forall r \in \mathbb{R},\ \text{a.e. } x \in \Omega : \quad |a(x, r)| \le \Lambda |r|^{p-1},$$
$$\forall r_1, r_2 \in \mathbb{R},\ r_1 \ne r_2,\ \text{a.e. } x \in \Omega : \quad (a(x, r_1) - a(x, r_2))(r_1 - r_2) > 0.$$

For convenience, we set $\phi = \varphi + x_2$, $u = \mathbf{p} + x_2$ and $g = 1 - \chi$. Now, we consider the following weak formulation of a nonlinear heterogeneous

1

evolution dam problem:

$$(\mathbf{P})\quad \begin{cases} \text{Find } (u,g) \in L^p(0,T;W^{1,p}(\Omega)) \times L^\infty(Q) \text{ such that :} \\ u \geq x_2,\ 0 \leq g \leq 1,\ g(u-x_2)=0 \quad \text{a.e. in } Q, \\ u = \phi \quad \text{on } \Sigma_2, \\ \displaystyle\int_Q \left[(a(x,u_{x_2}) - ga(x,1))\xi_{x_2} + g\xi_t\right] dxdt + \int_\Omega g_0(x)\xi(x,0)\,dx \leq 0 \\ \forall \xi \in W^{1,p}(Q),\ \xi = 0 \text{ on } \Sigma_3,\ \xi \geq 0 \text{ on } \Sigma_4,\ \xi(x,T)=0 \text{ for a.e. } x \in \Omega, \end{cases}$$

where g_0 is a function of the variable x such that

$$0 \leq g_0(x) \leq 1 \quad \text{a.e. } x \in \Omega.$$

If we replace Ω by an arbitrary bounded open of $\mathbb{R}^2$ and $(a(x,u_{x_2})-ga(x,1))\xi_{x_2}$ by $(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)).\nabla\xi$, where $e = (0,1)$ and $\mathcal{A}$ is an operator from $\Omega \times \mathbb{R}^2$ into $\mathbb{R}^2$ satisfying $x \mapsto \mathcal{A}(x,\xi)$ is measurable for all $\xi \in \mathbb{R}^2$, the function $\xi \mapsto \mathcal{A}(x,\xi)$ is continuous for a.e. $x \in \Omega$,

$$\forall \xi \in \mathbb{R}^2,\ \text{a.e. } x \in \Omega : \quad \lambda|\xi|^p \leq \mathcal{A}(x,\xi)\cdot\xi,$$
$$\forall \xi \in \mathbb{R}^2,\ \text{a.e. } x \in \Omega : \quad |\mathcal{A}(x,\xi)| \leq \Lambda|\xi|^{p-1},$$
$$\forall \xi,\eta \in \mathbb{R}^2,\ \xi \neq \eta,\ \text{a.e. } x \in \Omega : \quad (\mathcal{A}(x,\xi) - \mathcal{A}(x,\eta))\cdot(\xi-\eta) > 0,$$
$$\exists\, q > 1 \text{ such that } \operatorname{div}(\mathcal{A}(x,e)) \in L^q(\Omega),$$

the author in [1] established the existence of a solution by means of regularization using the Tychonoff fixed point theorem. In this work, we adapt the Poincaré inequality for our domain Ω and we apply such techniques as in [1] to prove the existence of a solution for the problem (P).

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ON THE MAXIMUM NUMBER OF LIMIT CYCLES OF A SECOND-ORDER DIFFERENTIAL EQUATION

SANA KARFES AND ELBAHI HADIDI

Abstract. This work concerns the qualitative study of a perturbed ordinary differential equation of second order. We study the limit cycles which can bifurcate from the center of the equation

$$\ddot{x} + x = 0. \tag{1}$$

2010 Mathematics Subject Classification. 34C25, 34C29.

Keywords and phrases. Periodic solution, averaging method, differential system.

1. Define the problem

In this work, we study the maximum number of limit cycles bifurcating from the periodic solutions of (1), when we perturb this equation as follows:

$$\ddot{x} + \varepsilon(1 + R^m(\theta))\psi(x, y) + x = 0,$$

where $\varepsilon > 0$ is a small parameter, m is an arbitrary non-negative integer, $\psi(x, y)$ is a polynomial of degree $n \geq 1$ and $\theta = \arctan(\frac{y}{x})$ and R is a trigonometric function. We determine an upper bound for the maximum number of limit cycles in equation (2) in the four cases where m and n are even and odd. The main tool used for proving this result is the averaging theory of first order.

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ON THE NUMERICAL SOLUTION OF FIRST ORDER HYPERBOLIC EQUATIONS ON SEMI-INFINITE DOMAINS

REMILI WALID AND RAHMOUNE AZEDINE

Abstract. This paper proposed a numerical method for solving first order hyperbolic equations on semi-infinite domains. The method of solution is based on the transformation of the original problem by means of a suitable mapping and one use the classical Jacobi polynomials collocation method to solve the mapped hyperbolic equation. This is by using the properties of Jacobi polynomials with the vec-operation and Kronecker product to reduce the hyperbolic equation to a system of linear algebraic equations with unknown Jacobi coefficients. Finally, some numerical examples are presented to illustrate the efficiency of the proposed method compared with other approaches.

2010 Mathematics Subject Classification. 45xx,65xx,34K28,39xx,42A10.

Keywords and phrases. Hyperbolic equations, A semi-infinite domains, Jacobi polynomials, A suitable mapping.

1. Define the problem

Partial differential equations (**PDEs**) on semi-infinite domains are encountered as model in many fields of science and engineering such as the earthquake engineering field and underwater acoustic problems. In this work, we consider the hyperbolic **PDEs** on semi-infinite domains of the form

$$\partial_t u(x,t) = a_1 \partial_x u(x,t) + a_2 u(x,t) + K(x,t), \quad x,t \in [0,\infty), \tag{1}$$

with the following condutions

$$u(0,t) = g_1(t), \qquad u(x,0) = g_2(x), \quad x,t \in [0,\infty), \tag{2}$$

where g_1, g_2,K are given sufficiently smooth functions and a_1, a_2 are constants whereas u is unknown function to be determined. The numerical solution of hyperbolic equation type (1) with conditions (2) have been discussed by many authors. For instance, the authors of [1] have used generalized Laguerre-gauss-radau scheme for solving E.q (1) with conditions (2). Exponential Jacobi-Galerkin method proposed from the authors of [2] to solve E.q (1) with conditions (2). The aim of this work is to extend the Jacobi polynomials for solving Eq. (1). This is by using algebraic and exponential mappings (see [3]) given by the following formulas, in which the constant $s > 0$ sets the length scale of the mappings

* Algebraic map

$$y = \theta_s(x) = \frac{x-s}{x+s}, \quad x = \psi_s(y) = s\left(\frac{1+y}{1-y}\right), \quad s > 0. \tag{3}$$

* Exponential map

$$y = \theta_s(x) = 1 - 2e^{-x/s}, \quad x = \psi_s(y) = -s\ln(\frac{1-y}{2}), \quad s > 0, \tag{4}$$

such that

$$y = \theta_s(x) = \psi_s^{-1}(x), \quad x \in \mathbb{R}^+ \text{ and } \frac{dx}{dy} = \theta_s'(y) > 0, \quad y \in (-1,1),$$
$$\theta_s(0) = -1, \quad \theta_s(+\infty) = 1 \text{ and } \psi_s(-1) = 0, \quad \psi_s(1) = +\infty.$$

The essential idea in our approach is to substitute x by $\psi(y,s)$ into the hyperbolic equation (1) and (2), and then applying the Jacobi polynomials collocation method to solve the resulting equation, wtich defined as

$$\begin{aligned}\partial_z u_s(y,z)\theta_s'(z) &= a_1\theta_s'(y)\partial_y u_s(y,z) \\ &+ a_2 u_s(y,z) + K_s(y,z), \quad y,z \in (-1,1),\end{aligned} \tag{5}$$

with the following condutions

$$u_s(-1,z) = g_1^s(z), \qquad u_s(y,-1) = g_2^s(y), \quad y,z \in (-1,1), \tag{6}$$

where

$$u_s(y,z) = u(\psi_s(y), \psi_s(z)) \quad , \quad K_s(y,z) = k(\psi_s(y), \psi_s(z)),$$
$$g_1^s(z) = g_1(\psi_s(z)) \quad , \quad g_2^s(y) = g_2(\psi_s(y)),$$

The classical Jacobi polynomials $J_n^{\alpha,\beta}(y)$ $(n \geq 0)$ are defined by (see[4])

$$(1-y)^\alpha(1+y)^\beta J_n^{\alpha,\beta}(y) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dy^n}\{(1-y)^{n+\alpha}(1+y)^{n+\beta}\}, \quad y \in (-1,1).$$

Let $w^{\alpha,\beta}(y) = (1-y)^\alpha(1+y)^\beta$ be the Jacobi weight function, for $\alpha, \beta > -1$. The Jacobi polynomials are mutually orthogonal in $L^2_{w^{\alpha,\beta}}(-1,1)$, i.e.,

$$(J_n^{\alpha,\beta}, J_m^{\alpha,\beta})_{w^{\alpha,\beta}} = \int_{-1}^{1} J_n^{\alpha,\beta}(y) J_m^{\alpha,\beta}(y) w^{\alpha,\beta}(y)dy = \gamma_n^{\alpha,\beta}\delta_{n,m}, \tag{7}$$

where $\delta_{n,m}$ is the Kronecker function, and

$$\gamma_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}. \tag{8}$$

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ON COMPUTATIONAL AND NUMERICAL SIMULATIONS OF THE RIEMANN PROBLEM FOR TWO-PHASE FLOWS CARBON DIOXIDE MIXTURES.

SOUHEYLA OUFFA AND DJAMILA SEBA

Abstract. In this work, we provide a computational simulations for the complete and exact solution to the Riemann problem for a one-dimensional two-phase carbon dioxide mixtures. Where the solution is obtained by solving the conservation of mass for each phase, the mixture conservation momentum equation and the mixture conservation energy equation of the two phases under conditions. And we present numerical simulations in conjuction with a computational simulations , it is Godunov's scheme which provides satisfactory results. numerical methods are provided to demonstrate the use of the exact framework and the proposed calculation.

2010 Mathematics Subject Classification. 35L65, 74S10, 35Lxx, 76Txx ,74Sxx.

Keywords and phrases. Conservation laws, finite volume methods, hyperbolic systems, two-phase flow, numerical methods.

1. Define the problem

In this paper, we study model describing a two-phase flow of a carbon dioxide mixture where we suggest a different solution for this model. The model is a system of coupled nonlinear hyperbolic differential equations. The purpose of this study is to provide a detailed presentation of the complete and accurate solution to the Riemann problem associated with the proposed model. The solution depends on conservation laws in a one-dimensional domain along with initial data separated by a single discontinuity, the solution is created under a set of suggestions and assumptions. Firstly we present the mathematical model that describes flow equations and illustrates Riemann's related problem. Then, we offer the elementary waves of the Riemann problem solution and build different waves based on the analytical solution. After that, description of solution strategy, and show a complete solution to the Riemann. then we extend the Godunov method in conjuction with the exact solution and the method of a second-order Godunov centred where the solution of the Riemann problem is fully numerical. Furthermore we conduct several tests with various problems dealing with shock and rarefaction waves to validate the presented analytical solution. Finally, we present the conclusion based on the results .

We obtain accurate analytical solutions based on the Riemann problem from the model equations program by main program that is being written. These are also evaluated numerically against the methods in which the Riemann solution is completely digital. Then We compare them .An excellent agreement was indicated between analytical results and numerical forecasts.

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ON A VISCOELASTIC WAVE EQUATION OF INFINITE MEMORY COUPLED WITH ACOUSTIC BOUNDARY CONDITIONS

ABDELAZIZ LIMAM, YAMNA BOUKHATEM,
AND BENYATTOU BENABDERRAHMAN

Abstract. This work deals with a coupled system of viscoelastic wave equation of infinite memory. The coupling is via by the acoustic boundary conditions. The semigroup theory is used to show the global existence of solution. Moreover, we investigate exponential stability of the system taking into account Gearhart-Prüss' theorem.

2010 Mathematics Subject Classification. 35A01, 74B05, 93D15.

Keywords and phrases. viscoelastic damping, global existence, exponential stability.

1. Define the problem

In this paper, we consider the following viscoelastic wave equation coupled with mixed boundary conditions

$$
(1) \quad \begin{cases}
u_{tt} - \operatorname{div}(\mathrm{A}\nabla u) + \int_0^{+\infty} g(s)\operatorname{div}(\mathrm{A}\nabla u(t-s))ds = 0 & \text{in} \quad \Omega \times \mathbb{R}_+ \\
u = 0 & \text{on} \quad \Gamma_0 \times \mathbb{R}_+ \\
\frac{\partial u}{\partial \nu_{\mathrm{A}}} - \int_0^{+\infty} g(t-s)\frac{\partial u}{\partial \nu_{\mathrm{A}}}(s)ds = z_t & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\
hz_{tt} + fz_t + mz + u_t = 0 & \text{on} \quad \Gamma_1 \times \mathbb{R}_+ \\
u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) & \text{for} \quad x \in \Omega \\
z(x,0) = z_0(x), \ z_t(x,0) = z_1(x) & \text{for} \quad x \in \Gamma_1,
\end{cases}
$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, such that Γ_0 and Γ_1 are closed and disjoint and $\nu = (\nu_1, \cdots, \nu_n)$ represents the unit outward normal to Γ, and the operator $\mathrm{A} = \Big(a_{ij}(x)\Big)_{1 \leq i,j \leq n}$.

The above model would be to describe the motion of fluid particles from rest in the domain Ω into part of the surface at a given point $x \in \Gamma_1$, which can be expressed by the pressure at that point. The relationship between the velocity potential $u_t = u_t(x,t)$ at a point on the surface and the normal displacement $z = z(x,t)$ is proportional to the pressure. It is called the acoustic impedance. This impedance may be complex in the case of the velocity potential was not in phase with the pressure. The coupling of our model (1) is via by the impenetrability boundary condition $(1)_3$ and the acoustic boundary condition $(1)_4$. Note that the term $\int_0^{+\infty} g(s)\operatorname{div}(\mathrm{A}\nabla u(t-s))ds$ is the infinite memory (past history) responsible for the viscoelastic damping, where g is called the relaxation function. The functions $h, f, m : \Gamma_1 \to \mathbb{R}^+$ are essentially bounded such that $h(x) \geq h_0$, $f(x) \geq f_0$ and $m(x) \geq m_0$ for a.e., and $u_0, u_1 : \Omega \to \mathbb{R}$, $z_0 : \Gamma_1 \to \mathbb{R}$ are given functions.

The partial differential equation (PDE) system of viscoelastic wave equation with acoustic boundary conditions was first introduced by Morse and Ingard [13] and developed by Beale [2], and Beale-Rosencrans [3]. In [2], the problem was formulated as an initial value problem in a Hilbert space and semigroup methods were used to solve it. The loss of decay has obtained by [2] provided that the term z_{tt} was included. Recently, the result concerning existence and asymptotic behavior of smooth, as well as weak solution of wave equation with acoustic boundary conditions have been established by many authors, see [6, 8, 11, 12]. Boukhatem and Benabderrahmane [5], studied the global existence and the exponential decay of solution of the system (1) in the absence of the second derivative z_{tt}. This absence brings us some difficulties in the study because of the abnormality of the system. It can not apply directly the semigroups theory or Faedo-Galerkin's procedure. They added in the argument the term εz_{tt} when $\varepsilon \to 0$ to overcome the difficulty.

The primary discussion touched upon by several authors is to use the integral term of relaxation function g instead of the damping term u_t. The question that have been focused their attention as an important works is the viscoelastic damping of memory effect should be strong enough to procreate the decay of the system.

One of important motivations to studying exponential stability of the associated semigroup comes from the spectral analysis. This purpose recalls the related results given by Gearhart-Prüss' theorem (see [9, 14]). It is shown that all eigenvalues approach a line that is parallel to the imaginary axis. Moreover, the resolvent operator is bounded for all eigenvalues of the generator associated. The proof is the combination of the contradiction argument with a PDE technique. Let us mention some papers on weakly dissipative coupled systems. In [10], the exponential decay is established of both wave equations are damped on the boundary. For weak damping acting only one equation, the lack of exponential decay to coupled wave equations was studied in [1, 7]. The authors obtained the optimal polynomial decay by using the recent result due to Borichev and Tomilov [4].

Our main result is devoted to study the global existence and the exponential stability of solution of (1), in which we analyze the spectral distribution in the complex plane. The semigroup theory is used to show the global existence of solution that its real part decreases with time. Motivated by the mentioned works above concerning Gearhart-Prüss' theorem, the exponential stability is concluded.

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ON SOLUTIONS OF BRATU-TYPE DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

ALI KHALOUTA

Abstract. Recently, nonlinear differential equations of fractional order (NDEFO) have attracted the attention of many researchers due to a wide range of applications in many fields of pure and applied mathematics such as: physics, fluid mechanics, electrochemistry, viscoelasticity, nonlinear control theory, nonlinear biological systems, hydrodynamics and other fields of science and engineering . In all these scientific fields, it is important to find exact or approximate solutions to these problems. There is therefore a marked interest in the development of methods for solving problems related to NDEFO. The exact solutions to these problems are sometimes too complicated to achieve by conventional techniques due to the complexity of the nonlinear parts involving them.

The aim of this talk is to present an analytical method called the general fractional residual power series method (GFRPSM) to find an analytical solution of a certain class of NDEFO in particular, Bratu-type differential equations of fractional order in the form

$$D^{\alpha}u(x)+\lambda\exp\left(u(x)\right)=0, 0<x<1, \lambda\in\mathbb{R}, \quad (1)$$

with the initial conditions

$$u(0)=a_0, u'(0)=a_1, \quad (2)$$

where D^{α} is the Caputo fractional derivative of order α, $1<\alpha\leq 2$.

2010 Mathematics Subject Classification. 34A08, 26A33, 34K28, 35C10.

Keywords and phrases. Bratu-type equation, Caputo fractional derivative, residual power series method, analytical solution.

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ON THE SOLUTION OF A CONTROL PROBLEM OF A VACCINE-CONTROLLED EPIDEMIC

BOUREMANI TOUFFIK AND BENTERKI DJAMEL

ABSTRACT. We use some recent developments in Dynamics Programming Method to obtain a rigorous solution of the optimal control problem formulated in [11] using Pontryagin's Maximum Principle. We use a certain refinement of Cauchy's Method of characteristics for stratified Hamilton-Jacobi equations to describe a large set of admissible trajectories and to identify a domain on which the value function exists and is generated by a certain admissible control and, its optimality is justified by the use of one of the well-known verification theorems as an argument for sufficient optimality conditions.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 49J15, 49L20, 34A60

KEYWORDS AND PHRASES. Optimal control, Differential inclusion, Pontryagin's maximum principle, Dynamic programming, Hamiltonian flow, Value function, Verification theorem.

1. INTRODUCTION

The aim of this paper is to apply step by step the dynamic programming theoretical algorithm, described in [5, 6] as well as, to combine these results with numerical procedures, to obtain a more rigorous and complete theoretically justified solution of the problem formulated as Example 7.3.10 in [11]. In fact, in [11], this problem is proposed to answer only a certain question, in the context of the using Pontryagin's Maximum Principle [1, 2, 3, 10, 11], but not to be studied in the rigorous manner in contrast to what we do below.

The importance of this theme comes to the fore with the unfortunate *COVID-19* pandemic and the need to discover its crypts by introducing it as dynamic models amenable to study, by using recent results in control theory.

2. POSITION OF THE PROBLEM

In [11] it has been considered a population affected by an epidemic, that is being tried to stop by vaccination. This leads us to solve the optimal control problem of *minimizing* the cost functional:

$$(1)\qquad \begin{cases} \mathcal{C}(u) = \alpha x_2(T) + \int_0^T u(t)^2\, dt, \\ x_1' = -r x_1 x_2 + u(t), \qquad x_1(0) = x_0^1 \\ x_2' = r x_1 x_2 - \gamma x_2 - u(t), \; x_2(0) = x_0^2 \\ x_3' = \gamma x_2, \qquad x_3(0) = x_0^3 \\ x_0 \in \mathbb{R}_+^3,\; x(T) = x_T,\; u(t) \in [0,a],\; t \in [0,T],\; T \text{ fixed}, \end{cases}$$

the functions involved have the following Medical virology significance:

For $t \in [0, T]$;

- $x_1(t)$: the number of non-infectious, but contaminable individuals,
- $x_2(t)$: the number of infectious individuals, who can infect others,
- $x_3(t)$: the number of individuals infected, and missing, or isolated from the rest of the population;
- $u(t)$: the vaccination rate (which is actually the control function),
- $r > 0$: the rate of infection;
- $\gamma > 0$: the disappearance rate.

2.1. **The dynamic programming formulation.** In order to the use the Dynamic Programming Approach in [5, 6], we reformulate the problem in (1) using standard notations in Optimal Control Theory and embedding this problem in a set of problems associated to each initial point in the phase-space as in [7, 8]. Thus, we obtain the following standard *Bolza* autonomous optimal control problem:

Given T, $a > 0$, find:

$$\inf_{u(.)} \mathcal{C}(y, u(.)), \ \forall y \in Y_0 \tag{2}$$

subject to:

$$\begin{array}{l} \mathcal{C}(y, u(.)) = g(x(T)) + \int_0^T f_0(x(t), u(t))\, dt, \\ x'(t) = f(x(t), u(t)),\ u(t) \in U(x(t))\ a.e.\ ([0,T]),\ x(0) = y, \\ x(t) \in Y_0,\ \forall t \in [0, T),\ x(T) \in Y_1,\ T \text{ fixed} \end{array} \tag{3}$$

defined by the following data:

$$\begin{array}{l} f(x, u) = (-rx_1x_2 + u, rx_1x_2 - \gamma x_2 - u, \gamma x_2),\ f_0(x, u) = u^2, \\ U(x) = U = [0, a],\ g(\xi) = \alpha \xi_2,\ \forall \xi = (\xi_1, \xi_2, \xi_3) \in Y_1, \\ Y_0 = \mathbb{R}_+^*,\ Y_1 = int(Y_1) \subset \mathbb{R}_+^*. \end{array} \tag{4}$$

The first main computational operation consists in the backward integration (for $t \leq 0$), of the Hamiltonian inclusion:

$$(x', p') \in d_S^{\#} H(x, p),\ (x(0), p(0)) = z = (\xi, q) \in Z_1^*, \tag{5}$$

defined by the generalized Hamiltonian field $d_S^{\#} H(.,.)$:

$$\begin{array}{l} d_S^{\#} H(x, p) = \{(x', p') \in T_{(x,p)} Z;\ x' \in f(x, \widehat{U}(x, p)), \\ < x', \overline{p} > - < p', \overline{x} >= DH(x, p)(\overline{x}, \overline{p}),\ \forall\ (\overline{x}, \overline{p}) \in T_{(x,p)} Z\}, \end{array} \tag{6}$$

and, the set of terminal transversality values defined in the general case by:

$$Z_1^* = \{(\xi, q) \in \overline{Z}, \xi \in Y_1,\ H(\xi, q) = 0,\ < q, \overline{\xi} >= Dg(\xi)\overline{\xi},\ \forall\ \overline{\xi} \in T_\xi Y_1\}. \tag{7}$$

As it is specified in the algorithm given in [5, 6], for each terminal point $z = (\xi, q) \in Z_1^*$ one should identify the maximal flows: $X^*(.) = (X(.), P(.)) : I(z) = (t^-(z), 0] \to Z$, of the Hamiltonian inclusion in (5) that satisfy the following admissibility conditions :

$$\begin{array}{l} X(t) \in Y_0,\ \forall\ t \in I_0(z) = (t^-(z), 0) \\ H(X(t), P(t)) = 0,\ \forall\ t \in I(z) \\ X'(t) = f(X(t), u(t)),\ u(t) \in \widehat{U}(X^*(t))\ a.e.\ I_0(z). \end{array} \tag{8}$$

The characteristic flow allow the identification of a subset of the set of initial states on which an optimal control and the corresponding value function are described, while the optimality is proved using a suitable Elementary Verification Theorem for value functions [2, 4, 5].

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Optimization of a PDE Problem and Application

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Abstract

In this paper we are interested in solving a hyperbolic PDE type, with some techniques based on the temporal discretization, the variational formulation and the intervention of the Lax-Milgram theorem. We have well answered the question of well-posed differential problems, likeable to approximation by known, reliable and compliant numerical methods such as finite elements.

The problem of the choice of our theme is the result of a reflection on the improvement of the approximation and the precision of the solution compared to the methods mentioned above.

Keywords

Hyperbolic problem, Optimization, The Galerkin-Newton method, Variational Formulation, PDE, finite element method.

1.Problem Definition

Let Ω a bounded open of $\mathbb{R}^n$, of a boudary Γ sufficiently regular and $0 \leq T \leq \infty$. Given a function

$$f : \Omega \longrightarrow \mathbb{R}^n,$$

Find the function

$$u : \Omega \times (0, T) \longrightarrow \mathbb{R}^n,$$

such as :

$$(P) \begin{cases} \frac{\partial^2}{\partial t^2}u(t,x) - \Delta u(t,x) - \frac{\partial}{\partial t}\Delta u(t,x) = f & (x,t) \in \Omega \times (0,T), \quad (1.1) \\ u(0,x) = 0 & on \ \Omega, \quad (1.2) \\ \frac{\partial}{\partial t}u(0,x) = 0 & on \ \Omega, \quad (1.3) \\ u \quad | \quad \Gamma \times (0,T) = 0, & (1.4) \\ \Delta u \ | \quad \Gamma \times (0,T) = 0, & (1.5) \end{cases}$$

Where f is a function given in $L^2(\Omega)$.

Our idea is presented by a contribution based on a combination of some known methods, where we involved some methods of optimizations, such as the method of energy minimization in the theoretical part, existence and uniqueness of the solution and method of Galerkin-Newton, in the numerical part and finally with an application. We achieved some satisfactory results!

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QUADRATIC DECOMPOSITION OF 2−ORTHOGONAL POLYNOMIALS SEQUENCES

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Abstract. In this work, we are interested in the quadratic decomposition of 2-monic orthogonal polynomials sequences ($2-MOPS$). We obtain the necessary and sufficient conditions for a monic polynomials sequence to be 2−orthogonal in terms of the sequences of the quadratic decomposition. Moreover, we obtain the links between the recurrence coefficients and the sequences of the quadratic decomposition. Also, we give the necessary and sufficient conditions for its principal components sequences to be orthogonal.

2010 Mathematics Subject Classification. 42C05, 33C45.

Keywords and phrases.. 2−Monic orthogonal polynomials ; Quadratic decomposition; 2−Symmetric $MOPS$..

1. Define the problem

from the quadratic decomposition of 2− orthogonal polynomials sequences generate new sequences and study their properties.

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STABILITY PROBLEM FOR AN EPIDEMIOLOGICAL MODEL (COVID-19)

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Abstract. The recent outbreak of the deadly and highly infectious COVID-19 disease caused by SARS-CoV-2 in Wuhan and other cities in China in 2019 has become a global pandemic as declared by World Health Organization (WHO) in the first quarter of 2020. In this work, our aim is to develop an SEIR mathematical model in order to minimize the number of the infected individuals and to study the impact of a control strategies. The proposed model is also studied in term of stability using the formula for calculating basic reproduction number R_0, and also an other approach to test stability of the model using state space approach is derived.

Keywords and phrases. Covid-19. SEIR model . Basic reproduction number. Stability. Optimal Control

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STATIONARY AND NON STATIONARY APPROXIMATIONS BY RBFS FOR SOLVING INTEGRAL AND PARTIAL DIFFERENTIAL EQUATIONS

DALILA TAKOUK AND REBIHA ZEGHDANE

Abstract. In this work, we give stationary and non stationary approximation by radial basis functions (RBFs) interpolation for solving integral and partial differential equations, The aim is to analyze the conflict between the theoritically achievable accuracy and numerical stability. The theoritical convergence rates may be difficult to achieve computationally due to the condition number of the resulting matrix growing with decreasing both fill distance and shape parameter. In this paper, we analyse the efficiency and applicability of the two approaches for scattered data approximation by globally and compactly supported RBFs, for solving some integral and partial differential equations. Some approximate solutions are considered by using numerical examples. Finally, some concluding remarks and ideas for future work are provided in the last section.

2010 Mathematics Subject Classification. 34K28, 35XX, 45XX35Q99, 65D05, 65Z99.

Keywords and phrases. Stationary and non stationary approximation, Radial basis functions, Shape parameter, Fill distance.

1. Introduction

Mesh methods gained much attention in recent years in mathematics and engineering community [1, 2, 3, 4, 5]. This meshfree discretisation techniques are based only on a set of independent points, therefore the costs of mesh generation is eliminated. It can be seen that this type of approximation provide a generation of numerical tools and it is more reliable than the traditional numerical methods such as finite element and difference methods which are limited to problems involving two or three parameters (space dimension). In this work, we present two sets of interpolation experiments with globally and compactly supported radial basis functions. We use non stationary approach to interpolation i.e, the support size remains fixed for increasingly danser sets of data sizes, on the other hand, we use the stationary approach, i.e, we scale the support size of the radial basis functions proportionally to the fill distance. We give a comparaison between the two approaches, and their computational complexity, for this, we take the gaussian, multiquadrics and some wendland's compactly supported radial basis functions to interpole some functions in one and two dimensional spaces. We use equally spaced grid in the unit square, and some nodes of orthogonal polynomials. The rate of convergence is determined, this rate is the exponent of the RMS error given by the formula rate= $\dfrac{ln(e_{k-1}/e_k)}{ln(h_{k-1}/h_k)}$, $k = 2, 3, ...$

where e_k is the error for experiment number k and h_k is the fill distance of the computional mesh. The problem of mesh approximation by radial basis functions is as follow. Radial basis function to scattered data $(x_i, f_i) \in \mathbb{R}^{n+1}$ for pairwise distincts points (centers) $x_1, x_2, \ldots, x_N, \in \mathbb{R}^n$, using a function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}$ to construct the interpolant

$$S(x) = \sum_{j=1}^{N} c_j \phi(\| x - x_j \|),$$

Via the linear system

$$S(x_i) = \sum_{j=1}^{N} c_j \phi(\| x_i - x_j \|) = f_i = f(x_i),\ 1 \leq i, j \leq N. \tag{1}$$

Which leads to the linear matrix system $AC = f$,

where $A = (a_{ij}) = \phi(\| x_i - x_j \|)$, $C = [c_1, c_2, \ldots, c_N]^T$, $f = [f_1, f_2, \ldots, f_N]^T$.

For wide choices of functions ϕ, the non singularity of the system (1) can be assured for the follwing radial basis functions

TABLE 1. Choices of ϕ for which the interpolation matrix is invertible.

Name	$\phi(r)$	
Gaussian	$e^{-\epsilon r^2}$	$\epsilon > 0$
Multiquadric	$\sqrt{r^2 + \epsilon^2}$	$\epsilon \geqslant 0$
Inverse Multiquadric	$\frac{1}{\sqrt{r^2 + \epsilon^2}}$	$\epsilon > 0$
Wendland's CSRBFs ($\Phi_{3,1}(r)$)	$(1 - r)_+^4(1 + 4r)$	$C^2 \cap PD_3$
Wendland's CSRBFs ($\Phi_{4,2}(r)$)	$(1 - r)_+^6(3 + 18r + 35r^2)$	$C^4 \cap PD_3$

We assume $F(r) = \phi(\sqrt{r})$ to be conditionally strictly positive definite of order zero [2], which implies that A is a positive definite matrix, so the problem is well posed i.e, there exists a unique solution if and only if A is invertible. In the univariate setting it is well knwon that one can interpolation to arbitrary data at N distinct sets, using a polynomial of degree $N - 1$, but for the multivariate settings, how even there is the negative results which due to curtis theorem (1959) [1] that there exist no Haar spaces of continuous functions except for one dimensional. For the use of shifts of one single basis function makes the radial basis approach particularly elegant and very attractive. This radial basis functions method depends on a shape parameter. So in this work we will clearly see the effects of the shape parameter on the condition number of the interpolation matrix therefore the numerical stability of our computations. Numerical studies, such as comparaison between the two approaches in the sense of accuracy and computational costs have been done, that illustrate the superior accuracy of each approach compared to solve both integral and partial differential equations.

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STUDY ON HOPF BIFURCATION FOR COMPRESSION QUASI-LINEAR SYSTEM,

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Abstract. Bifurcation analysis plays an important role in determining the phases of transition from an aerodynamic instability to another. This analysis is one of the main methods used for the study of nonlinear and quasi-linear systems for unsteady state. In our case, this study is applied to a compression model with an axial compressor. This model is developed from two principles: the first is the principle of movement at local equilibrium on the compressor and the second is based on the principle of mass balance of the plenum whose state functions of the system are the mass flow, mr, and pressure, Pp. A parametric study following eigenvalues made it possible to define the different domains of instability where a detailed set of conditions guarantees the existence of the Hopf bifurcation. A numerical simulation is presented to illustrate this analytical study.

2010 Mathematics Subject Classification. 34C23, 34D20, 35B35.

Keywords. Axial compressor, Hopf bifurcation, Surge and Rotating Stall.

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SOME RESULTS ON THE ASYMPTOTIC BEHAVIOUR OF SOME ANISOTROPIC SINGULAR PERTURBATION PROBLEMS

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ABSTRACT. The rate of asymptotic convergences in [1] is mentioned and shown far away from the boundary layers. In the present work we would given some boundary layer functions to get an asymptotic behaviour results on the whole domain for an anisotropic singular perturbation problems of an elliptic type.

KEYWORDS AND PHRASES. Anisotropic, singular perturbations, boundary layers, correctors, elliptic, boundary layer functions, rate of convergence.

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STABILITY ANALYSIS AND OPTIMAL CONTROL OF A FRACTIONAL-ORDER MODIFIED HIV/AIDS MODEL

NOUAR CHORFI, SALIM ZIDI, AND SALEM ABDELMALEK

Abstract. The fact that fractional-order models possess memory leads to modeling a fractional-order HiV/AIDS. We discuss the fractional order dynamics of HIV/AIDS model studied in [2]. We have divided the total population into five classes, namely (susceptible individuals, infective individuals who do not know that they are infected, HIV positive individuals who know that they are infected and that of the AIDS population). We prove that the proposed model has two distinct equilibria (disease-free equilibrium and the positive endemic equilibrium). Using the stability theorem, we establish the local stability of the disease-free equilibrium subject to the basic reproduction number being smaller than to unity, on the other hand, the endemic equilibrium subject to the basic reproduction being greater than unity. We have also discuss the previous model with three controls strategies of condom use u_1, screening of unawar infectives u_2 and treatment of unaware u_3.whose diagram is shown in Figure 1. Thus, the model is given by :

$$
\begin{aligned}
{}_0^cD_t^\alpha S(t) &= Q - \beta_m S - \mu S, \\
{}_0^cD_t^\alpha I_1(t) &= \beta_m \gamma S - (u_2\theta + \pi + \mu) I_1, \\
{}_0^cD_t^\alpha I_2(t) &= (1-\gamma)\beta_m S + u_2\theta I_1 - (\delta + u_3\kappa + \mu) I_2, \\
{}_0^cD_t^\alpha H(t) &= u_3\kappa I_2 - (\sigma\delta + \mu) H, \\
{}_0^cD_t^\alpha A(t) &= \pi I_1 + \delta I_2 + \sigma\delta H - (\beta + \mu) A.
\end{aligned}
$$

where

$$\beta_m = \frac{(1-u_1)\left(\beta_1 c_1 I_1 + \beta_2 c_2 I_2 + \beta_3 c_3 A\right)}{N}.$$

The controls strategies aimed at controlling of the spread of HIV/AIDS epidemic. The objective functional is defined as :

$$J(u_1, u_2, u_3) = \int_0^T \left(aI_1 + b_1u_1^2 + b_2u_2^2 + b_3u_3^2\right) dt.$$

Our aim here is to minimize the number of unaware infectives I_1, while minimizing the cost control u_{12}, u_2 and u_3. Then we seek an optimal control u_1^*, u_2^* and u_3^* such that

$$(u_1^*, u_2^*, u_3^*) = \min\left\{J(u_1, u_2, u_3) : u_1, u_2 \text{ and } u_3 \in U\right\},$$

where U is the admissible control set defined by

$$U = \left\{(u_1, u_2, u_3) : 0 \le u_i \le 1,\ t \in [0, T],\ \text{for } i = 1, 2, 3\right\}.$$

We give a general formulation for a FOCP and derive the necessary conditions for its optimality .

Finally, the numerical simulation using the Adams-type predictor corrector method to solve the fractional optimal control of the model, shows that this strategy helps to reduce the number of infected and the cost of control .

Keywords and phrases. HIV/AIDS model, Stability analysis, Fractional optimal control.

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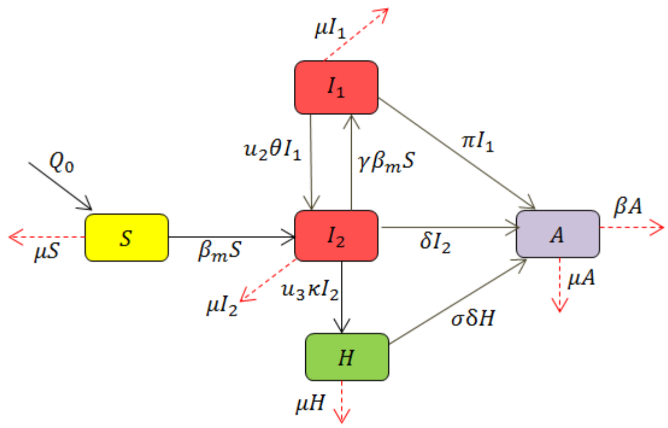

FIGURE 1. Flow Diagram of a fractional-order for the Modified HIV/AIDS disease transmission model with control.

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STABILIZATION OF FRACTIONAL ORDER CHAOTIC MODIFIED CHUA SYSTEM USING A STATE-FEEDBACK CONTROLLER

SAKINA BENRABAH AND SAMIR LADACI

Abstract. In this paper, we consider the problem of chaos stabilization for fractional order Chua's modified system with cubic nonlinearity. A state feedback controller is designed in order to force the system to converge to a stationary orbit. Simulation results are given to illustrate the effectiveness of the proposed control strategy.

2010 Mathematics Subject Classification. 26A33, 34H10, 93B52, 93D15.

Keywords and phrases. Fractional order system, Chaos, Chua modified system, Stabilization, State feedback control.

1. Define the problem

The fractional order form of Chua system is modelled as follows:

$$\begin{aligned} x^{(q)} &= a(y - x^3) + bx \\ y^{(q)} &= x - y - z \\ z^{(q)} &= cy \end{aligned} \tag{1}$$

By varying the total system order incrementally from 3.6 to 3.7, it is demonstrated that systems of "order" less than three can exhibit chaos as well as other nonlinear behavior. In particular, it presents a chaotic behavior for the parameters $q = 0.96$; $a = 10$; $b = 0.143$; $c = 16$, as stated by Hartley et al. [1] and Cafagna and Grassi [2].

Stability analysis of the chaotic system is studied for the fractional Chua chaotic system in closed loop by a linear state feedback given by,

$$U = KX \tag{2}$$

Where U is the control vector, K is a gain matrix, and X is the state vector. By an adequate adjustment of the gain K, we are able to stabilize the system on its stable orbits.

Numerical simulations are presented to show the effectiveness of the proposed fractional feedback method as shown in Figure 1, obtained for for $q = 0,96$ and $K = 0.58$.

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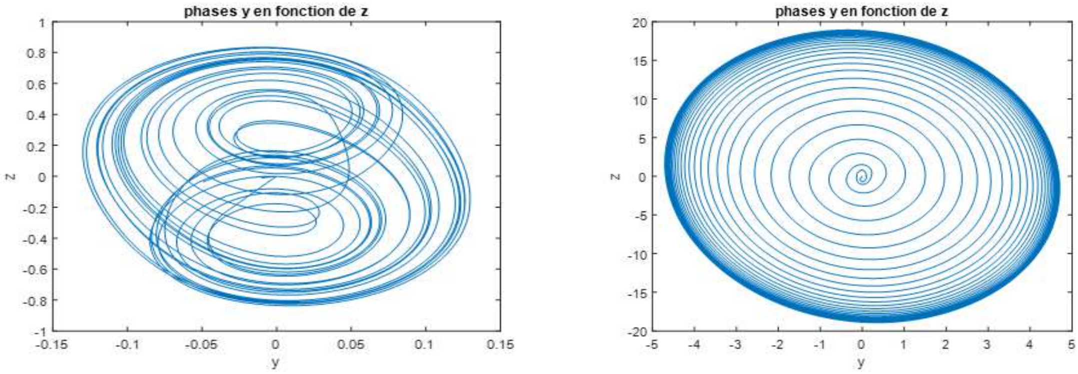

(a) (b)

FIGURE 1. Fractional order chaotic Chua system. (a) Without control. (b) with a state feedback control.

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THE LIMIT CYCLES OF TWO CLASSES OF CONTINUOUS PIECEWISE CUBIC DIFFERENTIAL SYSTEMS SEPARATED BY A STRAIGHT LINE

REBIHA BENTERKI

ABSTRACT. The main goal of this paper is to provide the maximum number of crossing limit cycles of continuous piecewise differential systems separated by the straight line $y = 0$ formed by a cubic isochronous center and an quadratic center. We prove that these piecewise differential systems can have at most two crossing limit cycles.

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2010 *Mathematics Subject Classification.* Primary 34C29, 34C25, 47H11.
Key words and phrases. limit cycles,continuous piecewise linear differential systems,linear centers, cubic isochronous centers.

The numerical analysis of Schwarz algorithm for a class of elliptic quasi variational inequalities

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Abstract

In this work, we study a Schwarz algorithm for a class of elliptic quasi-variational inequalities, where the obstacle depends to the solution.The author proved the error estimate in L^∞-norm for m subdomains with overlapping nonmatching grids using the geometrical convergence and the uniform convergence of variational inequalities.

Keywords: variational inequalities; Schwarz algorithm; finite element method; L^∞ error estimate

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THE NUMERICAL SOLUTION OF LARGE-SCALE DIFFERENTIAL T-RICCATI MATRIX EQUATIONS

LAKHLIFA SADEK, EL MOSTAFA SADEK, AND ALAOUI HAMAD TALIBI

ABSTRACT. In the present paper, we consider large-scale symmetric differential T-Riccati matrix equations. So far, it presents an unexplored problem in numerical analysis, theoretical results, and computational methods, which are lacking in the literature. we show how to apply the Krylov method such as the extended block Arnoldi algorithm to get low-rank approximate solutions. The initial problem is projected onto small subspaces to get low-dimensional symmetric differential equations that are solved using the Rosenbrok method. And report some numerical experiments to show the effectiveness of the proposed method for large-scale problems.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 65F50,15A24.

KEYWORDS AND PHRASES. extended block Arnoldi, Low-rank method, differential T-Riccati matrix equation, T-Sylvester equation, Rosenbrock method.

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UNIFORM CONVERGENCE OF MULTIGRID METHODS FOR VARIATIONAL INEQUALITIES

BELOUAFI MOHAMMED ESSAID, BEGGAS MOHAMED,
AND HAIOUR MOHAMED

Abstract. In this paper, we will apply the Multi-Grid Method for Variational Inequalities, in the measure where the obstacle depond of the solution. Moreover, we prove the uniforme convergence of this multi-grid algorithm with Gauss-Seidel's iteration as smoothing procedure.

Keywords and phrases. Variational inequality, Quasi-variational elliptic inequality, Multigrids methods, finite differences, finite element, Approximations.

1. Define the problem

1.1. **The continu problem.** Let Ω be an open in R^n, with sufficiently smooth boundary $\partial\Omega$ for $u, v \in H^1(\Omega)$, consider the bilinear form as follows:

$$(1.1)\quad a(u,v) = \int_\Omega [\sum_{1\leq i,j\leq n} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{1\leq i,j\leq n} a_i(x)\frac{\partial u}{\partial x_i} + a_0(x)u\cdot v]dx$$

- Where $a_{ij}(x), a_i(x), a_0(x), x \in \overline{\Omega}, 1 \leq i,j \leq n$ are sufficiently smooth coefficients and satisfy the following conditions:

$$\sum_{1\leq i,j\leq n} a_{ij}\psi_i\psi_j \geq v|\psi|^2, \psi \in R^n, v > 0$$

$$a_0(x) \geq \beta > 0,$$

-Where β is a constant.

We consider the following problem: Find $u \in H_0^1(\Omega)$ the solution of

$$(1.2)\qquad \begin{cases} a(u, v-u) \geq (f, v-u) & in\ \Omega, v \in H_0^1(\Omega) \\ u \leq \psi,\ v \leq \psi; & \psi \geq 0 \end{cases}$$

Were $f \in L^\infty(\Omega);\ f \geq 0,\ \psi \in W^{2,\infty}\ , tel\ que\ \psi > 0.$

1.2. **The discrete problem.** We denote by V_h the standard piecewise linear finite element space (where V_h form an internal approximation), we consider the discrete quasi-variational inequality Find $u_h \in V_h$ such that

$$(2.1)\qquad \begin{cases} a(u_h, v_h - u_h) \geq (f, v_h - u_h) & \forall u_h, v_h \in V_h \\ u_h \leq r_h\psi,\ v_h \leq r_h\psi \end{cases}$$

1

2. Description of the Multigrid Method for VIs

Let h_k be the discretization step over Ω. The finite element discretization conventionally leads to the discrete IV solution in finite dimension:

Find $u_k \in V_k$ such as:

$$\begin{cases} \langle A_k u_k, v_k - u_k \rangle \geq \langle f, v_k - u_k \rangle , & \forall v_k \in V_k \\ u_k \leq r_k \psi_k \ , & v_k \leq r_k \psi_k \end{cases} \tag{3.1}$$

put an iterated $u_k^v, v > 0$, we first determine $\bar{u}_k^v$ by p_k applications of a relaxation method

Note that:

$$\bar{u}_k^v = S_k^{p_k} \left(u_k^v \right) \tag{3.2}$$

or :

S_k is the iteration or smoothing operator

p_k is the number of iterations performed.

it is clear to verify that the IVs (7) are equivalent to the following PCNs.

Find $u_k \in \mathbb{R}^{n_k}$ solution of

$$\begin{cases} A_k u_k^* \leq F_k \ , \ u_k^* \leq r_k \psi_k \\ \left\langle A_k u_k^* - F_k \ , u_k^* - r_k \psi_k \right\rangle = 0 \end{cases} \tag{3.4}$$

Let us pose:

$$d_h^{(\nu)} = A_k \bar{u}_k^j - F_k, \quad \textit{le résidu de } u_k^v \tag{3.5}$$

It is immediate that the solution u_k^* of the problem (9) at the level k satisfies the following complementary problem:

$$\begin{cases} A_k u_k^* \leq A_k^v u_k - d_k^{(v)} \ , \ u_k^* \leq r_k \psi_k \\ \left\langle A_k u_k^* - A_k \bar{u}_k^v + d_k^{(v)} \ , u_k^* - r_k \psi_k \right\rangle = 0 \end{cases} \tag{3.6}$$

So to determine u_k completely, we need the calculate u_{k-1} at level $(k-1)$ as being the solution of:

$$\begin{cases} A_{k-1} u_{k-1} \leq g_{k-1} \ , \ u_{k-1} \leq r_k \psi_k \\ \left\langle A_{k-} u_{k-1} - g_{k-1} \ , u_{k-1} - r_k \psi_k \right\rangle = 0 \end{cases} \tag{3.7}$$

Where :

$$g_{k-1} = A_k r . \bar{u}_k^v - r . d_k^{(v)} \tag{3.8}$$

and r is the natural restriction

$$r = r_{k-1}^{-1} . r_k \tag{3.9}$$

we can interpret $u_{k-1} - r_k^{k-1} \bar{u}_k^v$ as an approximation at the level $k-1$ of the error $u_k^* - \bar{u}_k^\nu$.

Consequently, using an appropriate prolongation $p_{k-1}^{k} : R^{N_{k-1}} \rightarrow R^{N_k}$ we determine an improved iteration at the level k by

$$u_k^{v+1} = \bar{u}_k^{\nu} + p_{k-1}^{k}\left(u_{k-1} - r_k^{k-1}\bar{u}_k^{v}\right) \tag{3.10}$$

We are going to state a theorem of existence and unicity for the solution of the problems (1.2) and (2.1), and we will prove the convergence of Multigrid method for our problem (2.1).

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UZAWA METHODS FOR A LINEAR SYSTEM WITH DOUBLE SADDLE POINT STRUCTURE ARISING IN SHELL THEORY

KHENFER SAKINA AND MERABET ISMAIL

ABSTRACT. We consider Uzawa methods for solving a linear system with double saddle point structure arises in the finite element discretization of linear shell theory problems through the contact problem of a Naghdi's shells with a rigid obstacle in Cartesian coordinates.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 65M60, 65F10, 65N30.

KEYWORDS AND PHRASES. Naghdi's shell model, Uzawa's method, contact, finite elements.

1. Define the problem

The considered model is the unilateral contact of a shell with an obstacle which is actually one of the most currently used for numerical computations. The derivation of the model is based on the fundamental laws of elasticity and a priori geometrical assumptions which lead to constrained system. The resulting system is equivalent to a double mixed problem (i.e., a mixed problem with a double Lagrange multiplier) combining variational equalities and inequalities. The solution of the variational problem is sought in a functional space where some functional constrains must be satisfied. Unfortunatns can not be implemented in standard conforming way. Mixed methods can overcame this numerical difficulty efficiently. In this considered problem the mixed formulation is defined by mean of three bilinear forms. The continuous problem, written in mixed form takes the following form:

Find $(U, \psi, \lambda) \in \mathbb{X}(\omega) \times \mathbb{M}(\omega) \times \Lambda$ such that:

$$
(1) \qquad \begin{cases} \forall V \in \mathbb{X}(\omega), \; a_\rho(U,V) + b(V,\psi) - c(V,\lambda) = \mathcal{L}(V) \\ \quad \forall \chi \in \mathbb{M}(\omega), \qquad\qquad\qquad\qquad b(U,\chi) = 0 \\ \quad \forall \mu \in \Lambda, \qquad\qquad\qquad c(U, \mu - \lambda) \geq \langle \Phi, \mu - \lambda \rangle \end{cases}
$$

where, $\mathbb{X}(\omega)$, $\mathbb{M}(\omega)$ and Λ are three Hilbert spaces. The bilinear form a(,) is coercive on $\mathbb{X}(\omega)$ and (b + c)(,) satisfies the usual inf-sup condition. Standard finite element discretization of the variational problem leads to a large, sparse linear systems of equations of the form:

$$
(2) \qquad \begin{aligned} & AU + B^T\psi - C^T\lambda && = \mathcal{L} \\ & BU && = 0 \\ & (\mu - \lambda)^T CU \geq (\mu - \lambda)^T \phi && \end{aligned}
$$

where $A \in \mathbb{R}^{n\times n}$is symmetric positive definite (SPD),$B \in \mathbb{R}^{m\times n}$and $C \in \mathbb{R}^{p\times n}$ with $n \geq m + p$.

Throughout this work, Uzawa-type stationary methods is discussed. We present convergence results and eigenvalue bounds together.

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Abstract.The aim of this work is to study the optimal control strategy of a mathematical model of the COVID-19 transmission in the discrete case, and to investigate, in discrete time, optimal control strategie in which the controls are: quarantine and/or treatment. The studied population is divided into five compartments SII_rI_uR. Our objective is to find the best strategy to reduce the number of I. So, the Pontryagin's maximum principale, in discrete time, is used to characterize the optimal control. the numerical simulation is carried out using MATLAB. the obtaind results confirm the performance of the optimization strategy.
Key words: Optimal control
Discrete epidemec model
Quarantine,Treatment
Pontryagin's maximum principale.

Algebra and Geometry

ALMOST G-CONTACT METRIC STRUCTURES ON LIE GROUPS

BELDJILALI GHERICI, ALAYACH NOOR, AND BORDJI ABDELILLAH

Abstract. Starting from only a global basis of vector fields, we construct a class of almost contact metric structures and we give concrete example. Next, we investigate these structures on 3 and 5-dimensional Lie groups.

2010 Mathematics Subject Classification. 53C25, 53C15.

Keywords and phrases. Almost contact metric manifolds, Global basis. Lie Groups

1. Define the problem

Fortunately, the rich theory of vector spaces endowed with a Euclidean inner product can, to a great extent, be lifted to various bundles associated with a manifold. The notion of global (and local) frame plays an important technical role.
It should be mentioned however that a global basis of $\mathfrak{X}(M)$ (the Lie algebra of smooth vector fields on a manifold M) i.e., n vector fields that are linearly independent over $\mathcal{F}(M)$ and span $\mathfrak{X}(M)$, does not exist in general.

Manifolds that do admit such a global basis for $\mathfrak{X}(M)$ are called parallelizable. it is straightforward to show that a finite-dimensional manifold is parallelizable if and only if its tangent bundle is trivial (that is, isomorphic to the product, $M \times \mathbb{R}^n$).

As an illustration, we can prove that the tangent bundle, TS^1, of the circle, is trivial. Indeed, we can find a section that is everywhere nonzero, i.e. a non-vanishing vector field, namely

$$X(cos\theta, sin\theta) = (-sin\theta, cos\theta).$$

The reader should try proving that TS^3 is also trivial (use the quaternions). However, TS^2 is nontrivial, although this not so easy to prove.

More generally, it can be shown that TS^n is nontrivial for all even $n \geq 2$. It can even be shown that S^1, S^3 and S^7 are the only spheres whose tangent bundle is trivial. This is a rather deep theorem and its proof is hard.

Here, starting from a Global frame we construct a class of almost contact metric structures, specifically, many well-known almost contact metric structures (Sasakian, cosymplectic, Kenmotsu) and we confirm the construction each time with a concrete example showing that the case is nonvacuous. Next, we determine such structures on three and five-dimensional Lie algebras by direct calculation. We use the classification of three and five-dimensional Lie algebras given in [6].

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CRYPTOGRAPHY OVER THE ELLIPTIC CURVE $E_{a,b}(A_4)$ BY USING A PASSWORD

BILEL SELIKH

Abstract. We consider $A_4 := \mathbb{F}_{3^d}[\varepsilon] = \mathbb{F}_{3^d}[X]/(X^4)$ is a finite quotient ring, where $\varepsilon^4 = 0$ and $\mathbb{F}_{3^d}$ is the finite field of order 3^d with d be a positive integer. In this work, we introduce a diagram of cryptography based this ring. Firstly, we study the elliptic curve over this ring. Furthermore, we study the algorithmic properties by proposing effective implementations for representing the elements and the group law. Finally, we give an example cryptographic with a secret key.

2010 Mathematics Subject Classification. 94A60, 11T71, 11Y40, 14H52, 11T55.

Keywords and phrases.Cryptography, Elliptic Curves, Finite Rings, Finite Field.

1. Define the problem

The problem in this paper is based on a cryptographic application over the elliptic curves $E(A_4)$, so that we construct the subgroup G generated by a point P from $E(A_4)$. Next, we give each point of G a code and express it with a letter or symbol, and then define the encryption scheme by using a password, so that every message crypted is converted into a code and sent to the recipient.

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CLASSIFICATION OF MINIMAL SURFACES IN LORENTZ-HEISENBERG 3-DIMENSIONAL SPACE

BENSIKADDOUR DJEMAIA

Abstract. We first investigate the minimal translation surfaces i.e (surfaces with null mean curvature $H = 0$) in the 3-dimensional Lorentzian Heisenberg space $\mathcal{H}_3$ endowed with a left invariant metric g_1, we study six types of them. Then, we give the explicit expression of each type.

2010 Mathematics Subject Classification. 53A45, 53C20.

Keywords and phrases. Lorentzian Heisenberg 3− Space, Lorentzian metric, Translation surfaces, Minimal surfaces, Mean curvature.

1. Introduction

Lorentzian spaces, more precisely three dimensional Lie groups equipped with a left-invariant Lorentzian metric constitute the goal of several modern researches in pseudo-Riemannian geometry. The space $\mathcal{H}_3$, is a three dimensional Cartesian space with respect to the following product

$$(x, y, z) * (x', y', z') = (x + x',\ y + y',\ z + z' - xy'),$$

for any (x, y, z) and (x', y', z') of $\mathcal{H}_3$. The identity of this group is $(0, 0, 0)$ and the inverse of each element $(x, y, z) \in \mathcal{H}_3$ is $(-x, -y, -xy - z)$. Since a Lie group is a smooth manifold, we can endow it with a Riemannian metric. N. Rahmani and S. Rahmani proved in their article ([6]), that modulo an automorphism of the Lie algebra of the Heisenberg group $\mathcal{H}_3$ there exist three classes of left invariant Lorentzian metrics

$$\begin{aligned} g_1 &= -dx^2 + dy^2 + (xdy + dz)^2, \\ g_2 &= dx^2 + dy^2 - (xdy + dz)^2, \\ g_3 &= dx^2 + (xdy + dz)^2 - [(1 - x)\, dy - dz]^2. \end{aligned} \tag{1}$$

2. Main results

2.1. Minimal translation surfaces in $(\mathcal{H}_3, g_1)$.

In this section, we present some results on the characterization of the curvature of translation surfaces in the 3− dimensional Lorentzian Heisenberg space $\mathcal{H}_3$ endowed with the following left invariant Lorentzian metric

$$g_1 = -dx^2 + dy^2 + (xdy + dz)^2.$$

2.1.1. *Minimal surface equations in* $(\mathcal{H}_3, g_1)$. Let Σ be a surface in the Lorentzian Heisenberg 3−space $\mathcal{H}_1$ which represents the graph of the function $z = f(x, y)$, it is parameterized by

$$\begin{aligned} X:\ U \subseteq \mathbb{R}^2 &\to \mathbb{R}^3 \\ (x, y) &\mapsto (x, y, f(x, y)) \end{aligned}$$

1

Proposition 2.1. *The surface Σ defined above is a minimal surface in $3-$dimensional Lorentzian Heisenberg space $\mathcal{H}_3$ if and only if it's mean curvature H satisfies the following condition*

$$H = \frac{1}{2W^3}\left[(P^2-1)f_{yy} + (Q^2+1)f_{xx} - 2PQf_{xy} - PQ\right] = 0. \tag{2}$$

2.1.2. *Some types of minimal translation surfaces in $(\mathcal{H}_3, g_1)$.* A translation surface Σ (γ_1, γ_2) in $\mathcal{H}_3$ is a surface parameterized by

$$\begin{array}{cccc} X: & \Sigma & \rightarrow & \mathcal{H}_3 \\ & (x,y) & \mapsto & X(x,y) = \gamma_1(x) * \gamma_2(y) \end{array},$$

and obtained as a product of two generating not orthogonal curves γ_1 and γ_2 situated in the planes of coordinates of $\mathbb{R}^3$. Since the multiplication $*$ in the Lorentzian Heisenberg space is not commutative, then for each choice of curves γ_1 and γ_2 we may construct two translation surfaces, namely Σ (γ_1, γ_2) and Σ (γ_2, γ_1) which are different. In this section we define and study four types of translation surfaces in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_1)$.

2.1.3. *Surfaces of type 1 and 2.* Let the curves γ_1 and γ_2 be given by $\gamma_1(x) = (x, 0, g(x))$ and $\gamma_2(y) = (0, y, h(y))$, where g and h are two arbitrary surfaces.

Theorem 2.2. *The minimal translation surfaces Σ in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_1)$ of type 1 are parameterized by $X(x,y) = (x, y, g(x) + h(y) - xy)$ where*

$$g(x) = ax + x_0, \text{ with } a, x_0 \in \mathbb{R}$$

and

$$h(y) = \frac{c}{2}\left[(y-a)\sqrt{|(y-a)^2-1|} - \ln\left|(y-a) + \sqrt{|(y-a)^2-1|}\right|\right] + y_0, \text{ with } (c, y_0 \in \mathbb{R}).$$

Theorem 2.3. *The minimal translation surfaces Σ in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_1)$ of type 2 are parameterized by $X(x,y) = (x, y, g(x) + h(y))$ where*

$$h(y) = ay + b$$

and

$$g(x) = \frac{1}{2}\left[(x+a)\sqrt{(x+a)^2+1} + \sinh^{-1}(x+a)\right]$$

with a and b are real constants.

2.1.4. *Surfaces of type 3 and 4.* Let now the curves γ_1 and γ_2 be given by $\gamma_1(x) = (x, 0, g(x))$ and $\gamma_2(y) = (h(y), y, 0)$, where g and h are two arbitrary surfaces.

Theorem 2.4. *The minimal translation surface Σ of type 3 in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_1)$ are parameterized by*

$$X(x,y) = (x, 0, g(x)) * (h(y), y, 0) = (x + h(y), y, g(x) - xy)$$

where $g(x)$, and $h(y)$ are given by

Theorem 2.5. *The minimal translation surfaces Σ of type 4 in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_1)$ are parameterized by*

$$X(x,y) = (h(y), y, 0) * (x, 0, g(x)) = (x + h(y), y, g(x)),$$

where $h(y)$ is an affine function $h(y) = ay + b$, a and b are real constants such as $a \neq \pm 1$ and $g(x)$ is given by

- *if $x^2 + 1 - a^2 \geq 0$, then*

$$g(x) = \frac{1}{2}\left[x\sqrt{x^2+1-a^2} + \ln(x + \sqrt{x^2+1-a^2}) - ln(x + \sqrt{x^2+1-a^2})a^2\right] - \frac{a}{1-a^2}x^2,$$

- *if $x^2 + 1 - a^2 < 0$, then*

$$g(x) = \frac{1}{2}\left[x\sqrt{-x^2-1+a^2} - \tan^{-1}\frac{x}{\sqrt{-x^2-1+a^2}} + \tan^{-1}(\frac{x}{\sqrt{-x^2-1+a^2}})a^2\right] - \frac{a}{1-a^2}x^2.$$

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COMPLETE SYMMETRIC FUNCTIONS AND BIVARIATE MERSENNE POLYNOMIALS.

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Abstract. In this work, we give some new generating functions of bivariate Mersenne polynomials and the products of bivariate Mersenne polynomials with bivariate complex Fibonacci polynomials, bivariate complex Lucas polynomials, Jacobsthal and Jacobsthal Lucas numbers, Jacobsthal and Jacobsthal Lucas polynomials, and the products of bivariate Mersenne polynomials with Gaussian numbers and polynomials.

2010 Mathematics Subject Classification. Primary 05E05; Secondary 11B39.

Keywords and phrases.Symmetric functions, Generating functions, Bivariate Mersenne polynomials, Bivariate complex Fibonacci polynomials.

1. Define the problem

In this contribution, we shall define a useful operator denoted by $\delta^{k}_{e_1e_2}$ for which we can formulate, extend and prove new results based on our previous ones, see [4, 6, 7] In order to determine generating functions of bivariate Mersenne polynomials and the products of bivariate Mersenne polynomials with bivariate complex Fibonacci polynomials, bivariate complex Lucas polynomials, Jacobsthal and Jacobsthal Lucas numbers, Jacobsthal and Jacobsthal Lucas polynomials, and the products of bivariate Mersenne polynomials with Gaussian numbers and polynomials.

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COMPLETE HOMOGENEOUS SYMMETRIC FUNCTIONS OF BINARY PRODUCTS OF GAUSSIAN (p,q)-NUMBERS WITH MERSENNE LUCAS NUMBERS AT POSITIVE AND NEGATIVE INDICES

NABIHA SABA AND **ALI BOUSSAYOUD**

Abstract. In this work, we study the Mersenne Lucas numbers and some Gaussian (p,q)-numbers. We introduce a operator in order to derive some new symmetric properties of Gaussian (p,q)-Fibonacci and Gaussian (p,q)-Lucas numbers, Gaussian (p,q)-Pell and Gaussian (p,q)-Pell Lucas numbers. By making use of the operator defined in this work, we give some new generating functions for the products of Gaussian (p,q)-numbers with Mersenne Lucas numbers at positive and negative indices.

2010 Mathematics Subject Classification. Primary 05E05; Secondary 11B39.

Keywords and phrases. Mersenne Lucas numbers, Gaussian (p,q)-numbers, symmetric functions, generating functions.

1. Define the problem

In this contribution, we shall define a useful operator denoted by $\delta_{e_1 e_2}^{2-l}$ for which we can formulate, extend and prove new results based on our previous ones, see [5], [1] and [6]. In order to determine some new generating functions for the products of Gaussian (p,q)-Fibonacci numbers, Gaussian (p,q)-Lucas numbers, Gaussian (p,q)-Pell numbers, Gaussian (p,q)-Pell Lucas numbers with Mersenne Lucas numbers at positive and negative indices. In particular, the new generating functions of the products for Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell and Gaussian Pell Lucas numbers with Mersenne Lucas numbers at positive and negative indices are obtained.

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DERANGEMENT POLYNOMIALS WITH A COMPLEX VARIABLE

ABDELKADER BENYATTOU

ABSTRACT. In this paper, we define the derangement polynomials with a complex variable and we give some properties of these polynomials.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 11B83, 30C10.

KEYWORDS AND PHRASES. Derangement polynomials, Complex variable.

1. INTRODUCTION

Derangement polynomials are defined by

$$\mathcal{D}_n(x) = n! \sum_{k=0}^{n} \frac{(x-1)^k}{k!}.$$

It is clear that $\mathcal{D}_n(0)$ is the n-th derangement number, denoted by $\mathcal{D}_n$ counting the number of permutation of the set $[n] := \{1, ..., n\}$ without a fixed point. The exponential generating function for the derangement polynomials is

$$\sum_{n=0}^{\infty} \mathcal{D}_n(x) \frac{t^n}{n!} = \frac{e^{-t}}{1-t} e^{xt}. \tag{1}$$

For more information about these numbers and polynomials one can see [1, 2, 3, 4, 5].

If we replace x by z or $\overline{z}$ in (1), where

$$z = x + iy, \overline{z} = x - iy, i^2 = -1,$$

we get

$$\sum_{n=0}^{\infty} \mathcal{D}_n(z) \frac{t^n}{n!} = \frac{e^{-t}}{1-t} e^{(x+iy)t} = \frac{e^{-t}}{1-t} e^{xt} (\cos(yt) + i \sin(yt))$$

$$\sum_{n=0}^{\infty} \mathcal{D}_n(\overline{z}) \frac{t^n}{n!} = \frac{e^{-t}}{1-t} e^{(x-iy)t} = \frac{e^{-t}}{1-t} e^{xt} (\cos(yt) - i \sin(yt)).$$

If we add or subtract the identities presented above, we get

$$\sum_{n=0}^{\infty} [\mathcal{D}_n(z) + \mathcal{D}_n(\overline{z})] \frac{t^n}{n!} = \frac{2e^{-t}}{1-t} e^{xt} \cos(yt)$$

$$\sum_{n=0}^{\infty} [\mathcal{D}_n(z) - \mathcal{D}_n(\overline{z})] \frac{t^n}{n!} = \frac{2ie^{-t}}{1-t} e^{xt} \sin(yt).$$

Let $\mathcal{D}_{n,1}(z) = \mathcal{D}_n(z) + \mathcal{D}_n(\overline{z})$, and $\mathcal{D}_{n,2}(z) = \mathcal{D}_n(z) - \mathcal{D}_n(\overline{z})$, then we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{D}_{n,1}(z) \frac{t^n}{n!} &= \frac{2e^{-t}}{1-t} e^{xt} \cos(yt), \\ \sum_{n=0}^{\infty} \mathcal{D}_{n,2}(z) \frac{t^n}{n!} &= \frac{2ie^{-t}}{1-t} e^{xt} \sin(yt)\end{aligned}$$

That is now

$$\cos(yt) = \frac{e^{iyt} + e^{-iyt}}{2}, \quad \sin(yt) = \frac{e^{iyt} - e^{-iyt}}{2i},$$

then

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{D}_{n,1}(z) \frac{t^n}{n!} &= \frac{e^{-t}}{1-t} e^{xt} \left(e^{iyt} + e^{-iyt}\right) \\ &= \sum_{n=0}^{\infty} \mathcal{D}_n(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{[(iyt)^n + (-iyt)^n]}{n!} \\ &= \sum_{n=0}^{\infty} \mathcal{D}_n(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} (iy)^n (1 + (-1)^n) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_k(x) (iy)^{n-k} \left(1 + (-1)^{n-k}\right) t^n.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{D}_{n,1}(z) &= \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_k(x) (iy)^{n-k} \left(1 + (-1)^{n-k}\right), \\ \mathcal{D}_{n,2}(z) &= \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_k(x) (iy)^{n-k} \left(1 - (-1)^{n-k}\right).\end{aligned}$$

The derangement polynomials with a complex variable can be defined by

$$\mathcal{D}_n(z) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_k(x) (iy)^{n-k},$$

and we can write $\mathcal{D}_n(z)$ as follows

$$\mathcal{D}_n(z) = i^n \sum_{s=0}^{n} (-1)^s \binom{n}{2s} \mathcal{D}_{2s}(x) y^{n-2s} - i^{n+1} \sum_{s=0}^{n} (-1)^s \binom{n}{2s+1} \mathcal{D}_{2s+1}(x) y^{n-2s-1}.$$

The first few polynomials are:

$$\begin{aligned}\mathcal{D}_0(z) &= 1, \\ \mathcal{D}_1(z) &= x + iy, \\ \mathcal{D}_2(z) &= x^2 - y^2 + 1 + 2xyi, \\ \mathcal{D}_3(z) &= x^3 + 3x - 3xy^2 + 2 + i\left(-y^3 + 3x^2y + 3y\right).\end{aligned}$$

In particular, for $y = 0$ or $x = y = 0$, we have

$$\mathcal{D}_n(z) = \mathcal{D}_n(x), \quad \mathcal{D}_n(0) = \mathcal{D}_n.$$

2. Some properties of the derangement polynomials with a complex variable

Now we give some properties of the derangement polynomials with a complex variable

Lemma 2.1. *For any non-negative integer n, we have*

$$\mathcal{D}_n(z) = \sum_{k=0}^{n} (n)_k \left[\sum_{s=0}^{k} \frac{(x-1)^s}{s!}\right] (iy)^{n-k}.$$

where $(n)_k$ is the falling factorial defined by

$$(n)_k = n(n-1)\cdots(n-k+1) \text{ if } n \geq 1 \text{ and } (n)_0 = 1.$$

Proposition 2.2. *For any non-negative integer n there holds*

$$\begin{aligned}
\mathcal{D}_{n+1}(z) &= (n+1)\mathcal{D}_n(z) + (z-1)^{n+1}, \\
\mathcal{D}_{n+2}(z) &= (n+1)\left[\mathcal{D}_{n+1}(z) + \mathcal{D}_n(z)\right] + (z-1)^{n+1} + (z-1)^{n+2}.
\end{aligned}$$

The first few $\mathcal{D}_n(z)$ polynomials can be written as follows

$$\mathcal{D}_0(z) = 1, \mathcal{D}_1(z) = z, \mathcal{D}_2(z) = z^2 + 1, \mathcal{D}_3(z) = z^3 + 3z + 2.$$

Proposition 2.3. *Let z_0 and $z = z_0 + h$ be two points. The function $\mathcal{D}_n(z)$ is holomorphic on $\mathbb{C}$ and for any non-negative integer n, we have*

$$\begin{aligned}
\mathcal{D}'_n(z) &= n\mathcal{D}_{n-1}(z), \\
\mathcal{D}_n(z) &= \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_{n-k}(z_0)(z-z_0)^k.
\end{aligned}$$

If $z_0 = 0$, we obtain

$$\mathcal{D}_n(z) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_{n-k} z^k,$$

where $\mathcal{D}'_n(z)$ is the derivative of $\mathcal{D}_n(z)$.

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DAWN-SETS AND UP-SETS ON A TRELLIS STRUCTURE

SARRA BOUDAOUD AND LEMNAOUAR ZEDAM

Abstract. Down-sets and up-sets are two important families on any ordered set, they play a central role in its representation theory [1, 2, 5]. Also, they contribute to the construction of the concepts of ideal and its dual (a filter). In this talk, we introduce some families of sets associated with any finite set on a trellis structure (it is a structure like lattice $(L, \leq, \wedge, \vee)$ without the property of associativity of the operations $\wedge$ and $\vee$, which means also the elimination of the property of transitivity of the order relation $\leq$) [3, 4]). Further, we extend the notions of down-sets and up-sets to a trellis structure and discuss their various properties. We pay particular attention to the properties that remain valid and to those that are fails in absence of the (associativity) transitivity property.

Keywords and phrases. Down-set, Up-set, Pseudo ordered set, Trellis.

1. Define the problem

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FEUILLETAGES DU PLAN PROJECTIF COMPLEXE À ORBITES DE DIMENSION MINIMALE 6

par

Samir BEDROUNI & David MARÍN

Résumé. — L'ensemble $\mathbf{F}(d)$ des feuilletages de degré d du plan projectif complexe s'identifie à un ouvert de ZARISKI dans un espace projectif de dimension d^2+4d+2 sur lequel agit le groupe $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$. Dans cet exposé, nous présentons les grandes lignes de la démonstration d'un résultat de classification des feuilletages de $\mathbf{F}(d)$ à orbites de dimension minimale 6. Plus précisément, nous montrons qu'il y a exactement deux orbites $O(\mathcal{F}_1^d)$ et $O(\mathcal{F}_2^d)$ de dimension 6, nécessairement fermées dans $\mathbf{F}(d)$, ce qui généralise des résultats connus en degrés 2 et 3. Il s'agit de l'un des principaux résultats d'un article récent intitulé « Géométrie de certains feuilletages du plan projectif complexe », *cf.* arXiv:2101.11509.

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FUZZY IDEALS AND FILTERS ON A TRELLIS

SOHEYB MILLES AND LEMNAOUAR ZEDAM

Abstract. The purpose of this work is to investigate fuzzy ideal and fuzzy filter concepts on a trellis and their fundamental properties. We present interesting characterizations of these notions in terms of trellis operations and in terms of their specific subsets. Moreover, we introduce two interesting kinds, prime fuzzy ideals and prime fuzzy filters with respect the weakly associative meet and join operations of this trellis and investigate their various characterizations and properties.

2010 Mathematics Subject Classification. 03B52, 03G10, 06B10.

Keywords and phrases. Fuzzy set, Ideal, Filter, Trellis.

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GENERALIZED MULTIPLICATIVE $(\alpha;\beta)$-DERIVATIONS ON PRIME RINGS

MOHAMMADI EL HAMDAOUI AND ABDELKARIM BOUA

Abstract. Let R be an associative ring, U a non zero ideal of R, P a prime ideal of R and $G : R \to R$ is a multiplicative generalized (α, β)-derivation of R. In the present paper, we obtain description of the structure of R and information about the generalized (α, β)-derivation G which satisfies the following differential identities:
(**i**) $[G(x), G(y)] = [x, y]$, for all $x, y \in U$;
(**ii**) $G(x) \circ G(y) = x \circ y$, for all $x, y \in U$
(**iii**) $[G(x), G(y)] - [x, y] \in P$, for all $x, y \in R$;
(**iv**) $G(x) \circ G(y) - x \circ y \in P$, for all $x, y \in R$
(**v**) $[G_1(x), G_2(y)] \in P$, for all $x, y \in R$;
(**vi**) $G_1(x) \circ G_2(y) \in P$, for all $x, y \in R$;
(**vii**) $[G_1(x), y] + [x, G_2(y)]$, for all $x, y \in R$;
(**viii**) $G_1(x) \circ y + x \circ G_2(y)$, for all $x, y \in R$;
Finally, an example is given to demonstrate that the restrictions imposed on the hypothesis of our result are not superfluous.

Generalized multiplicative $(\alpha;\beta)$-derivations; SCP map; prime ring.

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GENERATING FUNCTIONS AND THEIR APPLICATIONS

ALI BOUSSAYOUD

Abstract. In this paper, we give some new generating functions of the products of Gaussian (p, q)-Fibonacci numbers, Gaussian (p, q)-Lucas numbers, Gaussian (p, q)-Pell numbers, Gaussian (p, q)-Pell Lucas numbers and (p, q)-modified Pell numbers with 2-orthogonal Chebyshev polynomials of the first kind and trivariate Fibonacci polynomials.
2010 Mathematics Subject Classification. 05E05; 11B39..
Keywords and phrases. Generating function, Gaussian (p, q)-Fibonacci numbers, Trivariate Fibonacci polynomials, Chebyshev polynomials.

1. Define the problem

Generating function were first introduced by Abraham de Moivre in 1730, in order to solve the general linear recurrence problem (see [5] Section 1.2.9, *Generating Functions*). One can generalize to formal power series in more than one indeterminate, to encode information about infinite multi-dimensional arrays of numbers.

This concept can be applied to solve many problems in mathematics. there is a huge chunk of mathematics concerning generating functions. It can be used to solve various kinds of counting problems easily, solve recurrence relations by translating the relation in terms of sequence to a problem about functions, prove combinatorial identities.

In simple words, generating functions can be used to translate problems about sequences to problems about functions which are comparatively easy to solve using maneuvers. (For more details, we can see [7, 4, 16]).

Given a sequence $(a_n)_{n\geq 0}$ of numbers (which can be integers, real numbers or even complex numbers) we try to describe the sequence in as simple a form as possible. Where possible, the best way is usually to express a_n as a function of n. Unfortunately, not all sequences can be described directly by such a formula, and in cases where they can, it is not always easy to find the formula. Therefore, in many cases we describe our sequence by a recurrence. Another way we could describe the sequence is to view the an as the coefficients of a formal power series $F(x) := \sum_{n=0}^{\infty} a_n x^n$, $F(x)$ is called the generating function of the sequence a_n.

Note that, we can define the exponential (or Hurwitz) generating function of a_n by

$$E(x) := \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

More generally, let $\Omega = (\omega_0, \omega_1, ...)$ be a sequence of nonzero real numbers. Then, following Comtet (see [9] p. 137), we define the ω-generating function of the sequence a_n by

$$\Omega(x) := \sum_{n=0}^{\infty} a_n x^n \omega_n.$$

Thus $F(x)$ and $E(x)$ are the special cases where $\omega_n = 1$ and $\omega_n = 1/n!$ respectively.

The literature on these topics is extremely vast. See further examples in [1, 3, 2, 15].

In this paper, we give the generating functions for the products of each following numbers sequences [10, 11]:

- *Gaussian* (p,q)-*Fibonacci numbers* $\{GF_{p,q,n}\}_{n\geq 0}$, defined recursively by,

$$\begin{cases} GF_{p,q,0} = i, \quad GF_{p,q,1} = 1, \\ GF_{p,q,n} = pGF_{p,q,n-1} + qGF_{p,q,n-2}, \; n \geq 2. \end{cases}$$

- *Gaussian* (p,q)-*Lucas numbers* $\{GL_{p,q,n}\}_{n\geq 0}$, defined by the recurrence relation,

$$\begin{cases} GL_{p,q,0} = 2 - ip, \quad GL_{p,q,1} = p + 2iq, \\ GL_{p,q,n} = pGL_{p,q,n-1} + qGL_{p,q,n-2}, \; n \geq 2. \end{cases}$$

- *Gaussian* (p,q)-*Pell numbers* $\{GP_{p,q,n}\}_{n\geq 0}$, defined by,

$$\begin{cases} GP_{p,q,0} = i, \quad GP_{p,q,1} = 1, \\ GP_{p,q,n} = 2pGP_{p,q,n-1} + qGP_{p,q,n-2}, \; n \geq 2. \end{cases}$$

- *Gaussian* (p,q)-*Pell Lucas numbers* $\{GQ_{p,q,n}\}_{n\geq 0}$ defined as follows,

$$\begin{cases} GQ_{p,q,0} = 2 - 2ip, \quad GQ_{p,q,1} = 2p + 2iq, \\ GQ_{p,q,n} = 2pGQ_{p,q,n-1} + qGQ_{p,q,n-2}, \; n \geq 2. \end{cases}$$

And

- (p,q)-*modified Pell numbers* $\{MP_{p,q,n}\}_{n\geq 0}$, given by,

$$\begin{cases} MP_{p,q,0} = 1, \quad MP_{p,q,1} = p, \\ MP_{p,q,n} = 2pMP_{p,q,n-1} + qMP_{p,q,n-2}, \; n \geq 2. \end{cases}$$

with the following polynomials sequences:

- the 2-*orthogonal monic Chebyshev polynomials* of the first kind (MPS) $\{\widehat{T}_n(x)\}_{n\geq 0}$, studied in [8], and defined by the following relation where α and γ are constants,

$$\begin{cases} \widehat{T}_0(x) = 1, \quad \widehat{T}_1(x) = x, \quad \widehat{T}_2(x) = x^2 - \alpha, \\ \widehat{T}_{n+3}(x) = x\widehat{T}_{n+2}(x) - \alpha\widehat{T}_{n+1}(x) - \gamma\widehat{T}_n(x), \; n \geq 0, \; \gamma \neq 0. \end{cases}$$

and

- the *trivariate Fibonacci polynomials*, introduced by E.G. Kocer and H. Gedikce in [6], and defined by the next relation,

$$\begin{cases} H_0(x,y,t) = 0, \quad H_1(x,y,t) = 1, \quad H_2(x,y,t) = x, \\ H_n(x,y,t) = xH_{n-1}(x,y,t) + yH_{n-2}(x,y,t) + tH_{n-3}(x,y,t), \; n \geq 3. \end{cases}$$

The technique applied here is based on the so-called *symmetric functions*.

The further contents of this paper are as follows. Section **??** gives some preliminaries that we will need in the sequel. More precisely, we present and prove our main result which relates the symmetric function with the symmetrizing operator $\delta_{e_1e_2}^{2-l}$. In section **??**, we give some new generating functions related to another Gaussian (p, q) numbers and 2-orthogonal Chebyshev polynomials. Section **??** is devoted to give some generating functions of the products of Gaussian (p, q) numbers with the trivariate Fibonacci polynomials.

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GENERATING FUNCTIONS OF PRODUCTS k-BALANCING NUMBERS, k-LUCAS BALANCING NUMBERS AND THE CHEBYSHEV

YAKOUBI FATMA AND **ALI BOUSSAYOUD**

Abstract. In this paper, we calculate the generating functions by using the concepts of symmetric functions. Although the methods cited in previous works are in principle constructive, we are concerned here only with the question of manipulating combinatorial objects, known as symmetric operators. The proposed generalized symmetric functions can be used to find explicit formulas....

2010 Mathematics Subject Classification. 05E05, 11b39.

Keywords and phrases. k-Balancing numbers; k-Lucas-balancing numbers; Generating functions; Chebyshev polynomials.

1. Define the problem

In this contribution, we shall define a useful operator denoted by $\delta^{k}_{a_1 a_2}$ for which we can formulate, extend and prove new results based on our previous ones([2, 3, 4]). In order to determine a new class of generating functions of binary products of some special numbers and polynomials, we combine between our indicated past techniques and these presented polishing approaches.

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GROUPS WHOSE PROPER SUBGROUPS OF INFINITE RANK ARE FINITE-BY-HYPERCENTRAL

AMEL DILMI

ABSTRACT. It is proved that if G is an $\mathfrak{X}$-group of infinite rank whose proper subgroups of infinite rank are finite-by-hypercentral groups, then so are all proper subgroups of G, where $\mathfrak{X}$ is the closure of the class of periodic locally graded groups by the closure operations $\acute{\boldsymbol{P}}$, $\grave{\boldsymbol{P}}$, $\boldsymbol{R}$ and $\boldsymbol{L}$.

This is a joint work with Nadir Trabelsi.

A group G is said to be of finite rank r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with a such property. If there is no a such r, then the group G is said to be of infinite rank. In recent years, many authors studied the structure of locally (soluble-by-finite) groups G of infinite rank in which every proper subgroup of infinite rank belongs to a given class $\mathfrak{Y}$ and they proved that all proper subgroups of G belong to $\mathfrak{Y}$, sometimes the group G itself belongs to $\mathfrak{Y}$ (see for instance, [2] and [3]). In particular, it is proved in [3, Theorem B'], that an $\mathfrak{X}$-group of infinite rank whose proper subgroups of infinite rank are locally nilpotent is itself locally nilpotent, where $\mathfrak{X}$ is the class introduced in [1] as the class obtained by taking the closure of the class of periodic locally graded groups by the closure operations $\acute{\boldsymbol{P}}$, $\grave{\boldsymbol{P}}$, $\boldsymbol{R}$ and $\boldsymbol{L}$. Clearly $\mathfrak{X}$ is a subclass of the class of locally graded groups that contains all locally (soluble-by-finite) groups. Recall that a group is said to be locally graded if every non-trivial finitely generated subgroup contains a proper subgroup of finite index. In [1], it is proved that an $\mathfrak{X}$-group of finite rank is almost locally soluble. Using [3, Theorem B'] and the fact that locally nilpotent groups of finite rank are hypercentral, one can see that an $\mathfrak{X}$-group of infinite rank whose proper subgroups of infinite rank are hypercentral has all its proper subgroups hypercentral. In the present work, we consider this problem for the class of finite-by-hypercentral groups and we prove the following result.

Theorem 1. *Let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are finite-by-hypercentral, then so are all proper subgroups of G.*

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Key words and phrases. Hypercentral, Locally (soluble-by-finite), rank.

GROUPS WITH RESTRICTIONS ON SOME SUBGROUPS GENERATED BY TWO CONJUGATES

FARES GHERBI AND NADIR TRABELSI

Abstract. Given a class of groups $\mathfrak{X}$, define $\bar{\mathrm{F}}\mathfrak{X}$ to be the class of groups G such that for every $x \in G$, there exists a normal subgroup $H(x)$ of finite index in G such that $\langle x, x^h \rangle \in \mathfrak{X}$ for every $h \in H(x)$. Let $\mathfrak{P}$ be the class of polycyclic groups, $\mathfrak{C}$ be the class of coherent groups and let $\mathfrak{M}\mathfrak{U}$ be the class of supersoluble extensions of groups satisfying the minimal condition on normal subgroups. In this paper, we prove that if G is a finitely generated soluble group in the class $\bar{\mathrm{F}}\mathfrak{P}$ (respectively, $\bar{\mathrm{F}}\mathfrak{C}$, $\bar{\mathrm{F}}(\mathfrak{M}\mathfrak{U})$), then it is polycyclic (respectively, coherent, finite-by-supersoluble).

2010 Mathematics Subject Classification. 20F16, 20F99.

Keywords and phrases. polycyclic groups, coherent groups, supersoluble groups, minimal condition on normal subgroups, finitely generated soluble groups.

1. Define the problem

Let $\mathfrak{X}$ be a class of groups; denote by $\mathrm{F}\mathfrak{X}$ the class of groups G such that for every $x \in G$, there exists a normal subgroup of finite index $H(x)$ such that $\langle x, h \rangle \in \mathfrak{X}$ for all $h \in H(x)$. The class $\mathrm{F}\mathfrak{X}$ was introduced in [1] and it was investigated for $\mathfrak{X}$ being the class $\mathfrak{N}_k$ of nilpotent groups of class at most the integer $k \geq 0$. Note that the class $\mathrm{F}\mathfrak{N}_1$ coincides with the class of FC-groups. The class $\mathrm{F}\mathfrak{X}$ was also studied in [2], where $\mathfrak{X}$ is respectively the class $\mathfrak{N}\mathfrak{F}$, $\mathfrak{F}\mathfrak{N}$ and $\mathfrak{T}\mathfrak{N}$ of nilpotent-by-finite, finite-by-nilpotent and periodic-by-nilpotent groups respectively. In this paper, we will consider a weaker version of the class $\mathrm{F}\mathfrak{X}$ for a couple of classes $\mathfrak{X}$ which are related to the class $\mathfrak{P}$ of polycyclic groups. More precisely, denote by $\bar{\mathrm{F}}\mathfrak{X}$ the class of groups G such that for every $x \in G$, there exists a normal subgroup of finite index $H(x)$ such that $\langle x, x^h \rangle \in \mathfrak{X}$ for all $h \in H(x)$. Note that if $\mathfrak{X}$ is a subgroup closed class, then we have that $\mathfrak{X} \subseteq \mathrm{F}\mathfrak{X} \subseteq \bar{\mathrm{F}}\mathfrak{X}$. The considération of the group $U(3, \mathbb{Z})$, of all 3×3 unitriangular matrices over $\mathbb{Z}$, which is torsion-free and nilpotent of class 2, shows that $\mathrm{F}\mathfrak{N}_1 \subsetneq \bar{\mathrm{F}}\mathfrak{N}_1$; so that, in general, $\mathrm{F}\mathfrak{X}$ is stictly smaller than $\bar{\mathrm{F}}\mathfrak{X}$. We will consider the class $\bar{\mathrm{F}}\mathfrak{X}$ where $\mathfrak{X}$ is respectively the class $\mathfrak{P}$, $\mathfrak{C}$ and $\mathfrak{M}\mathfrak{U}$ of polycyclic groups, coherent groups and supersoluble extensions of groups satisfying the minimal condition on normal subgroups. Recall that a group G is said to be coherent (respectively, supersoluble) if every finitely generated subgroup is finitely presented (respectively, if it has a finite normal series of cyclic factors). More precisely, we have shown the following results :

Theorem 1.1. *A finitely generated soluble group is in the class $\bar{F}\mathfrak{P}$ if, and only if, it is polycyclic.*

Theorem 1.2. *A finitely generated soluble group is in the class* $\bar{F}\mathfrak{C}$ *if, and only if, it is coherent.*

Theorem 1.3. *A finitely generated soluble group is in the class* $\bar{F}(\mathfrak{M}\mathfrak{U})$ *if, and only if, it is finite-by-supersoluble.*

Let $\mathbb{Q} = (\mathbb{Q}, +)$ be the additive group of rational numbers. Since $\mathbb{Q}$ is locally cyclic, it is in the classes $\bar{\mathrm{F}}\mathfrak{P}$ and $\bar{\mathrm{F}}(\mathfrak{M}\mathfrak{U})$. But $\mathbb{Q}$ is neither polycyclic nor finite-by-supersoluble, which shows that Theorem 1.1 and Theorem 1.3 are not true for all soluble groups.

In [3], Golod constructed an infinite 3-generated group G all of whose 2-generated subgroups are finite p-groups for the same prime p. So G is in classes $\bar{\mathrm{F}}\mathfrak{P}$ and $\bar{\mathrm{F}}(\mathfrak{M}\mathfrak{U})$; but G, as it is infinite, is neither polycyclic nor finite-by-supersoluble. Therefore Theorem 1.1 and Theorem 1.3 are not true for all finitely generated (residually finite) groups.

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HOMODERIVATIONS AND JORDAN RIGHT IDEALS IN 3-PRIME NEAR-RINGS

ABDELKARIM BOUA

Abstract. In the present paper, we study the commutativity of 3-prime right near-rings admitting homoderivations, which satisfy certain differential proprieties on a near-ring. Furthermore, examples are given to demonstrate that our hypotheses cannot be omitted

2010 Mathematics Subject Classification. Primary: 16N60. Secondary: 16W25, 16Y30.

Keywords and phrases. Near-rings, Jordan ideals, Homoderivations.

1. Define the problem

In this paper, $\mathcal{N}$ denotes a right near-ring with multiplicative center $Z(\mathcal{N})$; and usually $\mathcal{N}$ will be 3-prime, i.e. if for $x, y \in \mathcal{N}$, $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. A near-ring $\mathcal{N}$ is called zero-symmetric if $x0 = 0$ for all $x \in \mathcal{N}$ (recall that left distributivity yields $0x = 0$). Recall that $\mathcal{N}$ is called 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in \mathcal{N}$. For any pair of elements $x, y \in \mathcal{N}$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ will denote the well-known Lie product and Jordan product respectively. According to [9], an abelian near-ring $\mathcal{N}$ is a near-ring such that $(\mathcal{N}, +)$ is abelian. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$. An additive subgroup $\mathcal{J}$ of $\mathcal{N}$ is said to be a Jordan left (respectively right) ideal of $\mathcal{N}$ if $n \circ j \in \mathcal{J}$ (respectively $j \circ n \in \mathcal{J}$) for all $j \in \mathcal{J}$, $n \in \mathcal{N}$ and $\mathcal{J}$ is said to be a Jordan ideal of $\mathcal{N}$ if $j \circ n \in \mathcal{J}$ and $n \circ j \in \mathcal{J}$ for all $j \in \mathcal{J}$, $n \in \mathcal{N}$.

From the literature, a number of authors have studied commutativity theorems for prime or semiprime rings, $*$-prime rings and near-rings admitting derivation, generalized derivation, semiderivation, generalized semiderivation or two sided α-derivation satisfying the conditions: $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, $d([x, y]) = 0$, $d([x, y]) = [x, y]$, $d([x, y]) = x \circ y$, $d(x \circ y) = 0$, $d(x \circ y) = x \circ y$ for all $x, y \in \mathcal{N}$, for more details see the references [2], [3], [4], [5], [6], [7], [8], [11], [12], for example.

In [13] El Sofy (2000) defined a homoderivation on a ring $\mathcal{R}$ to be an additive mapping h from $\mathcal{R}$ into itself such that $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x, y \in \mathcal{R}$. An example of such a mapping is to let $h(x) = f(x) - x$ for all $x \in \mathcal{R}$ where f is an endomorphism on $\mathcal{R}$. It is clear that a homoderivation h is also a derivation if $h(x)h(y) = 0$ for all $x, y \in \mathcal{R}$. In this case, $h(x)\mathcal{R}h(y) = \{0\}$ for all $x, y \in \mathcal{R}$. So, if $\mathcal{R}$ is a prime ring, then the only additive mapping which is both a derivation and a homoderivation is the zero mapping.

In [13] El Sofy (2000) also proved the commutativity of prime rings admitting a homoderivation h that satisfies the condition $h([x,y]) = \pm[x,y]$ for all $x, y \in I$, where I is a two sided ideal of $\mathcal{R}$. Following this line of investigation, several authors studied homoderivations acting on appropriate subsets of the prime ring and $*$-prime rings. In [10] Asmaa Melaibari et al. studied the commutativity of rings admitting a homoderivation h such that $h([x,y]) = 0$ for all $x, y \in U$, where U is a nonzero ideal of $\mathcal{R}$. In [1] A. Al-Kenani et al. proved the commutativity of $*$-prime rings admitting homoderivations which commute with $*$ and satisfy the conditions: $h([x,y]) = 0$, $h(x \circ y) = 0$, $h([x,y]) = [x,y]$ and $h(x \circ y) = x \circ y$ for all $x, y \in I$, where I is a nonzero $*$-ideal of $\mathcal{R}$.

In this line of investigation, it is natural to ask if these results are still true if we replace prime rings and $*$-prime rings by 3-prime near-rings? In this direction, it is more interesting to study these type of identities by replacing the ring with a near-ring. The goal of the present paper is to study the structure of near-rings and also Jordan right ideals equipped with a new concept in near-rings called "homoderivations" which satisfying some algebraic conditions. In fact, our results generalize some results obtained in [13], [10] and [1]. Some new related results have also been obtained. Motivated by the concepts of Homoderivations on rings, here we initiate the concepts of Homoderivations on near-rings as follows:

Definition 1.1. *Let $\mathcal{N}$ be a near-ring. An additive mapping $h : \mathcal{N} \to \mathcal{N}$ is a homoderivation if $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x, y \in \mathcal{N}$.*

Note that the following example justifies the existence of homoderivations on a near-ring which not derivations.

Example 1.2. *Let $\mathcal{N} = (\mathbb{Z}/2\mathbb{Z}, +, .)$ such that " + " is the usual addition and "." the multiplicative law defined by $a.b = a$ for all $a, b \in \mathbb{Z}/2\mathbb{Z}$. Clearly $\mathcal{N}$ is a right near-ring which is not a ring and $h = id_{\mathcal{N}}$ is a homoderivation on $\mathcal{N}$. But not is a derivation on $\mathcal{N}$.*

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HARMONIC MAPS AND TORSE-FORMING VECTOR FIELDS

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Abstract. In this paper, we prove that any harmonic map from a compact orientable Riemannian manifold without boundary (or from complete Riemannian manifold) (M, g) to Riemannian manifold (N, h) is necessarily constant, with (N, h) admitting a torse-forming vector field satisfying some condition.

2010 Mathematics Subject Classification. 53C43, 58E20, 53A30.

Keywords and phrases. Harmonic maps, Bi-harmonic maps, Torse-forming vector fields.

1. Define the problem

Let (M, g) and (N, h) be two Riemannian manifolds, the energy functional of a map $\varphi \in C^\infty(M, N)$ is defined by

$$E(\varphi) = \int_M e(\varphi) v^g, \tag{1}$$

where $e(\varphi) = \frac{1}{2}|d\varphi|^2$ is the energy density of φ, $|d\varphi|$ is the Hilbert-Schmidt norm of the differential $d\varphi$ and v^g is the volume element on (M, g). A map $\varphi \in C^\infty(M, N)$ is called harmonic if it is a critical point of the energy functional, that is, if it is a solution of the Euler Lagrange equation associated to (1)

$$\tau(\varphi) = \operatorname{trace} \nabla d\varphi = \nabla^\varphi_{e_i} d\varphi(e_i) - d\varphi(\nabla^M_{e_i} e_i) = 0, \tag{2}$$

where $\{e_i\}$ is an orthonormal frame on (M, g), ∇^M is the Levi-Civita connection of (M, g), and ∇^φ denote the pull-back connection on $\varphi^{-1}TN$. Harmonic maps are solutions of a second order nonlinear elliptic system and they play a very important rôle in many branches of mathematics and physics where they may serve as a model for liquid crystal. One can refer to [6]-[8] for background on harmonic maps.

We shall consider a torse-forming vector field ξ, that is, a vector field which is always torse-forming along any curve traced in a Riemennian manifold (M, g) (see [11]-[14]). In this case, we have

$$\nabla^M_X \xi = fX + \omega(X)\xi, \quad \forall X \in \Gamma(TM), \tag{3}$$

for some smooth function f and 1-form ω on M, where ∇^M denotes the Levi-Civita connection of (M, g). The 1-form ω is called the generating form and the function f is called the conformal scalar. A torse-forming vector field ξ is called proper torse-forming if the 1-form ω is nowhere zero on a dense open subset of M. A torqued vector field is a torse-forming vector field ξ satisfying (3) with $\omega(\xi) = 0$ (see [3],[4]). In the case that ω is identically

zero, ξ is called a concircular vector field. In particular, if $\omega = 0$ and $f = 1$, then ξ is called a concurrent vector field. For the existence of torse-forming vector field on Riemannian manifold see for example [5] and [9].
A special torse-forming vector field or briefly a STF-vector field on a Riemennian manifold (M, g) is a torse-forming vector field ξ satisfying the equation (3) with generating form $\omega = \mu\xi^\flat$, for some smooth function μ on M, that is

$$\nabla^M_X \xi = fX + \mu g(X, \xi)\xi, \quad \forall X \in \Gamma(TM). \tag{4}$$

In the seminal work [10], where we proved that, if (M, g) is a compact Riemannian manifold without boundary, (N, h) is a Riemannian manifold, $\varphi : (M, g) \to (N, h)$ a harmonic map, assume that there is a proper homothetic vector field ξ on (N, h), that is $\mathcal{L}_\xi h = 2kh$, for some constant $k \in \mathbb{R}^*$, where $\mathcal{L}_\xi h$ is the Lie derivative of the metric h with respect to ξ. Then φ is a constant map. In the case of STF-vector field we obtain the following result; Let (M, g) be a compact orientable Riemannian manifold without boundary, and (N, h) be a Riemannian manifold admitting a STF-vector field ξ with conformal scalar f and generating form $\mu\xi^\flat$. If $f > 0$ and $\mu \geq 0$ on N, then any harmonic map φ from (M, g) to (N, h) is constant.

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TWO RECURRENT METHODS TO CONSTRUCT SEQUENCES OF IRREDUCIBLE POLYNOMIALS OVER $\mathbb{F}_{3^s}$ AND $\mathbb{F}_{5^t}$, RESPECTIVELY, OF DEGREE $4^k n$

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Abstract. In this paper, we consider the irreducibility of certain composite polynomials over $\mathbb{F}_q$, where $q = p^m$ and p is an odd prime. Then, we give two recurrent methods to construct sequences of irreducible polynomials over $\mathbb{F}_{3^s}$ and $\mathbb{F}_{5^t}$, respectively, of degree $4^k n$ $(k = 1, 2, \ldots)$.

2010 Mathematics Subject Classification. 12E05, 12E20

Keywords and phrases. finite field, polynomial composition, irreducible polynomial.

1. Define the problem

Let $\mathbb{F}_q$ be a finite field of q elements, where $q = p^m$ and p is a prime. Let $f(x)$ be a polynomial over $\mathbb{F}_q$ of degree $n \geq 1$. We say that $f(x)$ is *irreducible* over $\mathbb{F}_q$ if $f(x) = Q(x)S(x)$ with $Q(x), S(x) \in \mathbb{F}_q[x]$ implies that either $Q(x)$ or $S(x)$ is a constant polynomial.

Irreducible polynomials over finite fields are of great importance in both mathematical theory and practical applications, such as coding theory, cryptography, complexity theory, computer science and computational mathematics (see e.g., [4],[7],[10],[11]). This paper is devoted to the construction of irreducible polynomials of degree $4^k n$ $(k = 1, 2, \ldots)$ over finite fields of odd characteristics.

The following theorem is essential for us.

Theorem 1.1 (Cohen [3]). *Let $g(x), h(x) \in \mathbb{F}_q[x]$ be relatively prime polynomials. Let $f(x)$ be an irreducible polynomial over $\mathbb{F}_q$ of degree n. Then the composition*

$$h(x)^n f(\frac{g(x)}{h(x)})$$

is irreducible over $\mathbb{F}_q$ if and only if $g(x) - \alpha h(x)$ is irreducible over $\mathbb{F}_{q^n}$ for any root $\alpha \in \mathbb{F}_{q^n}$ of $f(x)$.

Remark. Many authors used Cohen's theorem in order to construct irreducible polynomials over $\mathbb{F}_q$ from a given irreducible polynomials over $\mathbb{F}_q$(see e.g., [1],[5],[6],[8],[9]). The main idea is to make the polynomial $g(x) - \alpha h(x)$ to be a known polynomial, say a binomial or trinomial.

The following theorem is the base of our results.

Theorem 1.2 (Dickson [2]). *Let $q = p^m$ where p is an odd prime. Let $f(x) = x^4 + ax^3 + bx^2 + cx + d \in \mathbb{F}_q[x]$, where $c = \frac{1}{2}ab - \frac{1}{8}a^3$. Then, $f(x)$ is irreducible over $\mathbb{F}_q$ if and only if $(\frac{1}{2}b - \frac{1}{8}a^2)^2 - d$ and $\frac{5}{16}a^4 - a^2 b + 16d$ are non-square in $\mathbb{F}_q$.*

Using Dickson's theorem, a construction by composition of irreducible polynomials over $\mathbb{F}_q$ of the form $P(g(x))$, where $P(x)$ is irreducible over $\mathbb{F}_q$ of degree n and $g(x) \in \mathbb{F}_q[x]$ is a polynomial of degree 4 has been established in [8]. We will use a similar method to give a construction of irreducible polynomials by composition of the form $h(x)^n P(g(x)/h(x))$, where $h(x) \in \mathbb{F}_q[x]$ is a polynomial of degree 4, that we did not find in any literature.

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ISODUAL QUASI-CYCLIC CODES OVER FINITE FIELDS

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AND T.AARON GULLIVER

Abstract. An isodual code is a linear code which is equivalent to its dual, and a self-dual code is a code which is equal to its dual. The class of isodual codes is important because it contains the self-dual codes as a subclass. In addition, isodual codes are contained in the larger class of formally self-dual codes, and they are of interest due to their relationship to isodual lattice constructions . Motivated by the numerous practical applications of code equivalency in code-based cryptography, we prove that two quasi-cyclic codes are permutation equivalent if and only if their constituent codes are equivalent. This gives conditions on the existence of isodual quasi-cyclic codes. These conditions are used to obtain isodual quasi- cyclic codes. Further, we provide a construction of isodual quasi-cyclic codes as the matrix product of isodual codes

Mathematics Subject Classication 94B05, 94B15, 94B60.

Keywords
Cyclic codes, Quasi-cyclic codes, Equivalence, Permutation group, Isod- ual codes, Self-dual codes.

1. we prove that two quasi-cyclic codes are permutation equivalent if and only if their constituent codes are equivalent. This gives conditions on the existence of isodual quasi-cyclic codes.

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KÄHLERIAN STRUCTURE ON THE PRODUCT OF TWO TRANS-SASAKIAN MANIFOLDS

HABIB BOUZIR AND GHERICI BELDJILALI

Abstract. It's shown that for some changes of metrics and structural tensors, the product of two trans-Sasakian manifolds is a Kählerian manifold. This gives new positive answer and more generally to Blair-Oubiña's open question. (See [1]). Concrete examples are given.

Mathematics Subject Classification (2010): 53C15 ; 53C40.

Keywords and phrases: Trans-Sasakian manifolds; Kählerian manifolds; product manifolds.

1. Define the problem

On the product of two almost contact manifolds, A. Morimoto [2] defined a natural almost complex structure (see (4.2) in this paper) and proved that this almost complex structure is integrable if and only if the two factors are normal almost contact manifolds. Later, M. Capursi [3] investigated almost Hermitian geometry of the product of two almost contact metric manifolds with the product metric, with respect to the almost complex structure defined by Morimoto. He shows that this product is Hermitian, Kählerian, almost Kählerian or nearly Kählerian, if and only if, the two factors are normal, cosymplectic, almost cosymplectic or nearly cosymplectic respectively.

Extending ideas from Capursi and Morimoto, Blair and Oubiña [1] considered conformal and related changes of the product metric with respect to a family of almost complex structures (see relation (3.1)) containing the one of Morimoto. Under the Kähler condition on the product manifold, Blair and Oubina proved that if one factor is Sasakian, the other is not, but that locally the second factor is of the type studied by Kenmotsu. The resuls are more general and given in terms of trans-Sasakian, α-Sasakian and β-Kenmotsu structures, finally they asked the open question: What kind of change of the product metric will make both factors Sasakian?

In [4], Watanabe survey almost Hermitian, Kähler, almost quaternionic Hermitian and quaternionic Kähler structures, naturally constructed on products of manifolds with almost contact metric and Sasakian structures and open intervals, as an application of these constructions. Next, he investigated almost Hermitian structures, naturally defined on the product manifolds of two almost contact metric and Sasakian manifolds, and asked some problems related to these topics.

In the same direction, Özdemir and al. [5], gave some properties that each factor should satisfy to make the almost Hermitian structure on the product manifold in a certain class of almost Hermitian manifolds.

Recently, in [6], we introduced the notion of generalized doubly D-homothetic bi-warping. we gave an application to some questions of the characterization of certain geometric structures. Our work has supported the point of view of the Calabi-Eckmann manifolds that the almost Hermitian structures defined on the product of two Sasakian manifolds are never Kählerian.

Here, again we based on the open question of Blair-Oubiña (see [1],[4]), but with a new techniques which requires a change in the two directions, metrics and structural tensors of the two Trans-Sasakian manifolds, which gave a positive response to the question.

This paper is organized in the following way:
Section 2, is devoted to the background of the structures which will be used in the sequel. In Section 3, we introduce a new deformation of almost contact metric structure and we give some geometric properties. Section 3 is devoted to the construction of a class of interesting examples in dimension 3. In the last section, we focus on our main goal where we construct Kählerian manifold using the product of two Trans-Sasakian manifolds with a concrete example.

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MORE ON FUZZY TOPOLOGIES GERERATED BY FUZZY RELATIONS

KHEIR SAADAOUI

Abstract. We study fundamental properties of the notion of fuzzy topology generated by fuzzy relation given by Mishra and Srivastava. Some necessary examples are given. Moreover, we teat the lattice structure of a family of fuzzy topologies generated by fuzzy relations and we study necessary structural characteristics of this lattice.

2010 Mathematics Subject Classification.06B30, 03E72, 03F55

Keywords and phrases. Fuzzy set, topology, relation.

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NOTE ON A THEOREM OF ZEHNXIAG ZHANG

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Abstract. A sequence of strictly positive integers is said to be primitive if none of its terms devide the others. In this paper, we give a new proof of a result, conjectured by P. Erdős and Z. Zhang in 1993 [3], on a primitive sequence whose the number of the prime factors of the termes counted with multiplicity is at most 4. The objective of this proof is to improve the complexity, which helps to prove this conjecture.

2010 Mathematics Subject Classification. Primary 11Bxx.

Keywords and phrases. Primitive Sequence, Prime Number, Erdős Conjecture.

1. Define the problem

Attempt to prove the conjecture d'Erdős over primitive sequences on the sum $\sum 1/(a \log a)$.

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Naturally Harmonic Maps Between Tangent Bundles

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Mots-clés : Horizontal lift; vertical lift; Natural metrics; tangent map; harmonic map

Abstract

In this paper, we investigate the harmonicity of a tangent map $\phi : (TM, \tilde{g}) \longrightarrow (TN, \tilde{h})$, in the case where the tangent bundles TM, TN are endowed with a natural Riemannian metrics $\tilde{g}, \tilde{h}$. In this work we generalise previous results connecting to article A. Sanini (see [12]) .

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NEW LDPC CODES

BENNENNI NABIL

Abstract. In this article we have defined a new family of LDPC code over finite chain ring$R$ of four elements, we have modified several methods of the construction. Using the Gray application we obtained quasi-cyclic LDPC codes of the index 2 and we generalized this result in the finite chain rings of n elements such that we obtained quasi-cyclic codes of the index n.

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Cyclic codes over ring, new LDPC codes, several methods of the construction LDPC codes.

1. Define the problem

The first systematic and algebraic construction of LDPC codes based on finite geometries was proposed by Kou, Lin and Fossorier in the 2000s [10], [11], [12], [6], [13]. The LDPC class of finite geometry has a good minimum distance and the Tanner graphs do not have short cycles. Their structure is cyclic or quasi-cyclic, so that their encoding is simple and can be realized with linear shift registers. With this type of codes of great length, we obtain a very good error performance. The construction and decoding of LDPC codes can be done in several ways. An LDPC code is characterized by its parity matrix.

In this article we have defined a new family of LDPC code over finite chain ring$R$ of four elements, we have modified several methods of the construction. Using the Gray application we obtained quasi-cyclic LDPC codes of the index 2 and we generalized this result in the finite rings of n elements such that we obtained quasi-cyclic codes of the index n.

this paper is organized as follows. In section 2 we present some preliminaries of finite chain ring R and the cyclic code in this ring. In section 3 we defined a new family of non binary LDPC code, such that we have using the Gary map, we have obtained the binary regular code LDPC code. In section 4 we give several method for construction of LDPC code over finite chain ring R.

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On The Normality Of Toeplitz Matrices

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Abstract

In this paper, we study the normal structure of powers of normal Toeplitz matrices in the finite state. Every finite complex normal Toeplitz matrix T is one of following structures:
(Type I) : a rotation and a translation of a Hermitian Toeplitz matrix **(Type I)**, that is $T = \alpha I + \beta \mathcal{H}$, where α and β are complex numbers, and $\mathcal{H}$ is a Hermitian Toeplitz; or
(Type II) : is a generalised circulant which means a Toeplitz matrix of the form

$$T = \begin{pmatrix} a_0 & a_N e^{i\theta} & \ddots & a_1 e^{i\theta} \\ a_1 & a_0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_N e^{i\theta} \\ a_N & \ddots & a_1 & a_0 \end{pmatrix}$$

for some fixed real θ.
Our work consists in studying T^n $n \in \mathbb{N}$ and seeing whether it remains of the same type as T, be it (**I**) or (**II**).

Keywords : Normal Toeplitz martix, Hermetian matrix, Circulant matrix, Power.

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On Translation Surfaces with Zero Gaussian Curvature in Sol_3 Space

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Abstract

In this work we classified translation invariant surfaces with zero Gaussian curvature in the 3−dimensional Sol group .

Key words and phrases: Flat Surfaces, Homogeneous Space.
Mathematics Subject Classifications (2010): 49Q20. 53C22.

1 Introduction and Preliminaries

During the recent years, there has been a rapidly growing interest in the geometry of surfaces in the homogeneous space Sol_3 focusing on minimal and constant mean curvature and totally umbilic surfaces. This was initiated by R.Souam and E.Toubiana [18, 19], and by R.Lopez and M.I.Munteanu [8, 9] . More general many works are devoted to studying the geometry of surfaces in 3-homogeneous space Sol_3. See for example [10],[7],[12],[4],[13].
The Sol_3 geometry is eight models geometry of Thurston, see [21] .It is a Lie group endowed with a left-invariant metric, it is a homogeneous simply connected 3−manifold with a 3−dimensional isometry group, see [2].It is isometric to $\mathbb{R}^3$ equipped with the metric

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$

where (x, y, z) the usual coordinates of $\mathbb{R}^3$.
The group structure of Sol_3 is given by

$$(x', y', z') \star (x, y, z) = (e^{-z'}x + x', e^{z'}y + y', z + z').$$

The isometries are

$$(x, y, z) \mapsto (\pm e^{-c}x + a, \pm e^{c}y + b, z + c)$$

and

$$(x, y, z) \mapsto (\pm e^{-c}y + a, \pm e^{c}x + b, -z + c).$$

where a, b end c are any real numbers.
A left-invariant orthonormal frame $\{E_1, E_2, E_3\}$ in Sol_3 is given by

$$E_1 = e^{-z}\frac{\partial}{\partial x},\ E_2 = e^{z}\frac{\partial}{\partial y},\ E_3 = \frac{\partial}{\partial z}.$$

The Levi-Civita connexion $\widetilde{\nabla}$ of Sol_3 with respect to this frame is

$$\begin{aligned} \widetilde{\nabla}_{E_1}E_1 = -E_3,\ \widetilde{\nabla}_{E_1}E_2 = 0,\ \widetilde{\nabla}_{E_1}E_3 = E_1, \\ \widetilde{\nabla}_{E_2}E_1 = 0, \widetilde{\nabla}_{E_2}E_2 = E_3, \widetilde{\nabla}_{E_2}E_3 = -E_2 \\ \widetilde{\nabla}_{E_3}E_1 = 0,\ \widetilde{\nabla}_{E_3}E_2 = 0,\ \widetilde{\nabla}_{E_3}E_3 = 0. \end{aligned} \tag{1.1}$$

2 Flat Translation Surfaces in Sol$_3$

2.1

In this section we classified complete flat translation surfaces (Σ) in Sol_3 which are invariant under the one parameter group of isometries $(x, y, z) \mapsto (x, y + c, z)$.Clearly,such a surface is generated by a curve γ in the totally geodesic plane $\{y = 0\}$.Discarding the trivial case of a vertical plane $\{x = x_0\}$, we can assume that γ is locally is a graph over the $x-$axis.Thus γ is given by $\gamma(x) = (x, 0, z(x))$. Therefore the generated surface is parameterized by

$$X(x, y) = (x, y, z(x)),\ (x, y) \in \mathbb{R}^2.$$

We have an orthogonal pair of vector fields on (Σ), namely,

$$e_1 := X_x = (1, 0, z') = e^z E_1 + z' E_3.$$

and

$$e_2 := X_y = (0, 1, 0) = e^{-z} E_2.$$

The coefficients of the first fundamental form are:

$$E =< e_1, e_1 >= z'^2 + e^{2z},\ F =< e_1, e_2 >= 0,\ G =< e_2, e >= e^{-2z}.$$

As a unit normal field we can take

$$N = \frac{-z'e^{-z}}{\sqrt{1 + z'^2 e^{-2z}}} E_1 + \frac{1}{\sqrt{1 + z'^2 e^{-2z}}} E_3$$

The covariant derivatives are

$$\widetilde{\nabla}_{e_1} e_1 = 2z'e^z E_1 + (z'' - e^{2z}) E_3$$

$$\widetilde{\nabla}_{e_1} e_2 = -z'e^{-z} E_2$$

$$\widetilde{\nabla}_{e_2} e_2 = e^{-2z} E_3.$$

The coefficients of the second fundamental form are

$$l =< \widetilde{\nabla}_{e_1} e_1, N >= \frac{-2z'^2 + z'' - e^{2z}}{\sqrt{1 + z'^2 e^{-2z}}}$$

$$m =< \widetilde{\nabla}_{e_1} e_2, N >= 0$$

$$n =< \widetilde{\nabla}_{e_2} e_2, N >= \frac{e^{-2z}}{\sqrt{1 + z'^2 e^{-2z}}}.$$

Let K_{ext} be the extrinsic Gauss curvature of (Σ),

$$K_{ext} = \frac{ln - m^2}{EG - F^2} = \frac{-2z'^2 e^{-2z} + z'' e^{-2z} - 1}{(1 + z'^2 e^{-2z})^2}. \tag{2.1}$$

In order to obtain the intrinsic Gauss curvature K_{int}, recall that $K_{int} = K_{ext} + K(e_1 \wedge e_2)$, where $K(e_1 \wedge e_2)$ is the sectional curvature of each tangent plane spanned by e_1 and e_2, and

$$K(e_1 \wedge e_2) = \frac{\langle R(e_1, e_2)e_2, e_1 \rangle}{< e_1, e_1 >< e_2, e_2 > - < e_1, e_2 >^2}$$

where

$$R(e_1, e_2)e_2 = \widetilde{\nabla}_{e_1} \widetilde{\nabla}_{e_2} e_2 - \widetilde{\nabla}_{e_2} \widetilde{\nabla}_{e_1} e_2 - \widetilde{\nabla}_{[e_1,e_2]} e_2$$

Now we easily compute

$$\widetilde{\nabla}_{e_1} \widetilde{\nabla}_{e_2} e_2 = e^{-z} E_1 - 2z' e^{-2z} E_3$$

$$\widetilde{\nabla}_{e_2} \widetilde{\nabla}_{e_1} e_2 = -z' e^{-2z} E_3$$

$$\widetilde{\nabla}_{[e_1,e_2]} e_2 = 0.$$

Thus we have

$$K(e_1 \wedge e_2) = \frac{1 - z'^2 e^{-2z}}{1 + z'^2 e^{-2z}}.$$

Consequently, the intrinsic Gauss curvature is

$$K_{int} = \frac{e^{-2z}[z'' - 2z'^2 - z'^4 e^{-2z}]}{(1 + z'^2 e^{-2z})^2}. \tag{2.2}$$

So that (Σ) is a flat surface in Sol_3 if and only if

$$K_{int} = 0,$$

that is, if and only if

$$z'' - 2z'^2 - z'^4 e^{-2z} = 0 \tag{2.3}$$

to classify flat surfaces must solve the equation (2.3)
We note that for z equal to a constant ($z = z_0$) is a solution of the equation (2.3).
If z is not constant ($z' \neq 0$),suppose that $z' = p$, and

$$z'' = \frac{dp}{dx} = \frac{dp}{dz}\frac{dz}{dx} = p.p'(z)$$

equation (2.3) becomes

$$p.p' = 2p^2 + p^4 e^{-2z}.$$

or

$$p^{-3}.p' = 2p^{-2} + e^{-2z}. \tag{2.4}$$

and suppose that $p^{-2}=h$, equation (2.4) becomes

$$\frac{-1}{2}h' = 2h + e^{-2z}. \tag{2.5}$$

homogeneous solutions of equation (2.5) is

$$h(z) = K.e^{-4z}.$$

and general solutions of the equation (2.5) is

$$h(z) = e^{-4z}(a - e^{2z}),$$

where $a \in \mathbb{R}^{*,+}$ and $z \in]-\infty, \ln(\sqrt{a})[$.Therefore

$$p(z) = \pm\frac{1}{\sqrt{h(z)}} = \pm\frac{e^{2z}}{\sqrt{a-e^{2z}}}.$$

and we have

$$z' = \pm\frac{e^{2z}}{\sqrt{a-e^{2z}}}.$$

or

$$\frac{dz}{dx} = \pm\frac{e^{2z}}{\sqrt{a-e^{2z}}}$$

so separating variables, we obtain

$$\int dx = \int \pm\frac{\sqrt{a-e^{2z}}}{e^{2z}}dz$$

i.e

$$x = \pm\int \frac{\sqrt{a-e^{2z}}}{e^{2z}}dz + \alpha,$$

where $\alpha \in \mathbb{R}$.
we substitute $\tanh(t) = \frac{\sqrt{a-e^{2z}}}{\sqrt{a}}$, $dz = -\tanh(t)dt$, and $e^{2z} = \frac{a}{\cosh^2(t)}$, therefore

$$\int \frac{\sqrt{a-e^{2z}}}{e^{2z}}dz = -\frac{-1}{\sqrt{a}}\int \sinh^2(t)dt = -\frac{1}{8\sqrt{a}}[e^{2t} - e^{-2t}] + \frac{t}{2\sqrt{a}},$$

and as $t = arc\tanh\left(\frac{\sqrt{a-e^{2z}}}{\sqrt{a}}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{\sqrt{a-e^{2z}}}{\sqrt{a}}}{1-\frac{\sqrt{a-e^{2z}}}{\sqrt{a}}}\right)$, thus

$$\int \frac{\sqrt{a-e^{2z}}}{e^{2z}}dz = \frac{1}{2\sqrt{a}}arc\tanh\left(\frac{\sqrt{a-e^{2z}}}{\sqrt{a}}\right) - \frac{1}{8\sqrt{a}}\left[\left(\frac{\sqrt{a}+\sqrt{a-e^{2z}}}{\sqrt{a}-\sqrt{a-e^{2z}}}\right) - \left(\frac{\sqrt{a}-\sqrt{a-e^{2z}}}{\sqrt{a}+\sqrt{a-e^{2z}}}\right)\right]$$

and is calculated by the following

$$\int \frac{\sqrt{a-e^{2z}}}{e^{2z}}dz = \frac{1}{2\sqrt{a}}arc\tanh\left(\frac{\sqrt{a-e^{2z}}}{\sqrt{a}}\right) - \frac{\sqrt{a-e^{2z}}}{2e^{2z}}.$$

Therefore

$$x(z) = \pm\left(\frac{1}{2\sqrt{a}}arc\tanh\left(\frac{\sqrt{a-e^{2z}}}{\sqrt{a}}\right) - \frac{\sqrt{a-e^{2z}}}{2e^{2z}}\right) + \alpha.$$

As conclusion, we have

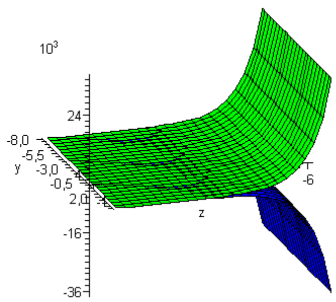

Figure 1: Non extendable flat surface in Sol_3 :$x(z) = \pm\left(\frac{1}{2\sqrt{0.2}}arc\tanh\left(\frac{\sqrt{0.2-e^{2z}}}{\sqrt{0.2}}\right)-\frac{\sqrt{0.2-e^{2z}}}{2e^{2z}}\right)+2, a=0.2,\ z=-6..-1,,\ y=-4..8$.

Theorem 2.1. •*The only non extendable flat translation surfaces in Sol_3 which are invariant under the one parameter group of isometries $(x,y,z)\mapsto(x,y+c,z)$, are the surfaces whose parametrization is $X(x,y)=(x,y,z(x))$ where x and z satisfy*

$$x=\pm\left(\frac{1}{2\sqrt{a}}arc\tanh\left(\frac{\sqrt{a-e^{2z}}}{\sqrt{a}}\right)-\frac{\sqrt{a-e^{2z}}}{2e^{2z}}\right)+\alpha.$$

where $a\in\mathbb{R}^{*,+}$, $\alpha\in\mathbb{R}$ and $z\in]-\infty,\ln(\sqrt{a})[$.
•*In particular the only complete flat translation surfaces in Sol_3 which are invariant under the one parameter group of isometries $(x,y,z)\mapsto(x,y+c,z)$, are the planes $z=z_0$.*

Theorem 2.2. •*The only complete extrinsically flat translation surfaces in Sol_3 which are invariant under the one parameter group of isometries $(x,y,z)\mapsto(x,y+c,z)$, are parametrized by*

$$X(x,y)=\left(x,y,\ln\left(\frac{1}{\sqrt{-x^2+2\lambda x+\mu}}\right)\right),$$

where $\lambda,\mu\in\mathbb{R}$, and $\lambda^2+2\mu>0$ $\alpha\in\mathbb{R}$ and $x\in]\lambda-\sqrt{\lambda^2+2\mu},\lambda+\sqrt{\lambda^2+2\mu}[$.

Proof. We know that Σ is extrinsically surface if and only if $K_{ext}=0$, and we have $K_{ext}=0$ equivalent to

$$2z'^2e^{-2z}-z''e^{-2z}=-1.$$

we remark that $2z'^2e^{-2z} - z''e{-}2z = (-z'e^{-2z})'$, thus

$$-z'e^{-2z} = -x + \lambda, \tag{2.6}$$

where $\lambda \in \mathbb{R}$,and we integrate the equation 2.6

$$z(x) = \ln\left(\frac{1}{\sqrt{-x^2 + 2\lambda + 2\mu}}\right),$$

where $\mu \in \mathbb{R}$, and $\lambda^2 + 2\mu > 0$ $\alpha \in \mathbb{R}$ and $x \in]\lambda - \sqrt{\lambda^2 + 2\mu}, \lambda + \sqrt{\lambda^2 + 2\mu}[$.

□

2.2

In this section we classified complete flat translation surfaces (Σ) in Sol_3 which are invariant under the one parameter group of isometries $(x, y, z) \mapsto (x + c, y, z)$.Clearly,such a surface is generated by a curve β in the totally geodesic plane $\{x = 0\}$.Discarding the trivial case of a vertical plane $\{y = y_0\}$, we can assume that β is locally is a graph over the $y-$axis.Thus β is given by $\beta(y) = (0, y, z(y))$. Therefore the generated surface is parameterized by

$$X(x, y) = (x, y, z(y)),\ (x, y) \in \mathbb{R}^2.$$

We have an orthogonal pair of vector fields on (Σ), namely,

$$e_1 := X_x = (1, 0, 0) = e^z E_1.$$

and

$$e_2 := X_y = (0, 1, z') = e^{-z}E_2 + z'E_3.$$

The coefficients of the first fundamental form are:

$$E =< e_1, e_1 >= e^{2z},\ F =< e_1, e_2 >= 0,\ G =< e_2, e_2 >= z'^2 + e^{-2z}.$$

As a unit normal field we can take

$$N = \frac{-z'e^z}{\sqrt{1 + z'^2e^{2z}}}E_2 + \frac{1}{\sqrt{1 + z'^2e^{2z}}}E_3$$

The covariant derivatives are

$$\widetilde{\nabla}_{e_1}e_1 = -e^{2z}E_3,$$
$$\widetilde{\nabla}_{e_1}e_2 = z'e^zE_1,$$
$$\widetilde{\nabla}_{e_2}e_2 = -2z'e^{-z}E_2 + (z'' + e^{-2z})E_3.$$

The coefficients of the second fundamental form are

$$l =< \widetilde{\nabla}_{e_1}e_1, N >= \frac{-e^{2z}}{\sqrt{1 + z'^2e^{2z}}}$$

$$m =< \widetilde{\nabla}_{e_1}e_2, N >= 0$$

$$n = < \widetilde{\nabla}_{e_2} e_2, N > = \frac{2z'^2 + z'' + e^{2z}}{\sqrt{1 + z'^2 e^{2z}}}.$$

Let K_{ext} be the extrinsic Gauss curvature of (Σ),

$$K_{ext} = \frac{ln - m^2}{EG - F^2} = \frac{-2z'^2 e^{2z} - z'' e^{2z} - 1}{(1 + z'^2 e^{2z})^2}. \tag{2.7}$$

In order to obtain the intrinsic Gauss curvature K_{int}, recall that $K_{int} = K_{ext} + K(e_1 \wedge e_2)$, where $K(e_1 \wedge e_2)$ is the sectional curvature of each tangent plane spanned by e_1 and e_2, and

$$\begin{aligned} K(e_1 \wedge e_2) &= \tfrac{\langle R(e_1,e_2)e_2,e_1 \rangle}{<e_1,e_1><e_2,e_2> - <e_1,e_2>^2} \\ &= \tfrac{R_{1212} + z'^2 R_{1313}}{1 + z'^2 e^{2z}} \\ &= \tfrac{1 - z'^2 e^{2z}}{1 + z'^2 e^{2z}}. \end{aligned}$$

Consequently, the intrinsic Gauss curvature is

$$K_{int} = \frac{e^{2z}[z'' + 2z'^2 + z'^4 e^{2z}]}{(1 + z'^2 e^{2z})^2}. \tag{2.8}$$

So that (Σ) is a flat surface in Sol_3 if and only if

$$K_{int} = 0,$$

that is, if and only if

$$z'' + 2z'^2 + z'^4 e^{2z} = 0 \tag{2.9}$$

to classify flat surfaces must solve the equation (2.9)
We note that for z equal to a constant ($z = z_0 \in \mathbb{R}$) is a solution of the equation (2.9).
If z is not constant ($z' \neq 0$),suppose that $z' = q$, and

$$z'' = \frac{dq}{dx} = \frac{dq}{dz}\frac{dz}{dx} = q.q'(z)$$

equation (2.9) becomes

$$q.q' = -2q^2 - q^4 e^{2z}.$$

or

$$q^{-3}.q' = -2q^{-2} - e^{2z}. \tag{2.10}$$

and suppose that $q^{-2} = g$, equation (2.10) becomes

$$\frac{-1}{2} g' = -2g - e^{2z}. \tag{2.11}$$

homogeneous solutions of equation (2.11) is

$$g(z) = K.e^{4z}.$$

and general solutions of the equation (2.11) is

$$g(z) = e^{4z}(a - e^{-2z}),$$

where $a \in \mathbb{R}^{*,+}$ and $z \in]\ln(\sqrt{a}), +\infty[$. Therefore

$$q(z) = \pm\frac{1}{\sqrt{g(z)}} = \pm\frac{e^{-2z}}{\sqrt{a - e^{-2z}}}.$$

and we have

$$z' = \pm\frac{e^{-2z}}{\sqrt{a - e^{-2z}}}.$$

or

$$\frac{dz}{dy} = \pm\frac{e^{-2z}}{\sqrt{a - e^{-2z}}}$$

so separating variables, we obtain

$$\int dy = \int \pm\frac{\sqrt{a - e^{-2z}}}{e^{-2z}}dz$$

i.e

$$y = \int \pm\frac{\sqrt{a - e^{-2z}}}{e^{-2z}}dz + \delta,$$

where $\delta \in \mathbb{R}$.
we substitute $\tanh(t) = \frac{\sqrt{a-e^{-2z}}}{\sqrt{a}}$, $dz = \tanh(t)dt$, and $e^{-2z} = \frac{a}{\cosh^2(t)} = a(1-\tanh^2(t)$, therefore

$$\int \frac{\sqrt{a - e^{-2z}}}{e^{-2z}}dz = -\frac{1}{\sqrt{a}}\int \sinh^2(t)dt = \frac{1}{8\sqrt{a}}[e^{2t} - e^{-2t}] - \frac{t}{2\sqrt{a}},$$

and as $t = arc\tanh\left(\frac{\sqrt{a-e^{-2z}}}{\sqrt{a}}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{\sqrt{a-e^{-2z}}}{\sqrt{a}}}{1-\frac{\sqrt{a-e^{-2z}}}{\sqrt{a}}}\right)$, thus

$$\int \frac{\sqrt{a - e^{-2z}}}{e^{-2z}}dz = -\frac{1}{2\sqrt{a}}arc\tanh\left(\frac{\sqrt{a - e^{-2z}}}{\sqrt{a}}\right)+\frac{1}{8\sqrt{a}}\left[\left(\frac{\sqrt{a} + \sqrt{a - e^{-2z}}}{\sqrt{a} - \sqrt{a - e^{-2z}}}\right) - \left(\frac{\sqrt{a} - \sqrt{a - e^{-2z}}}{\sqrt{a} + \sqrt{a - e^{-2z}}}\right)\right]$$

and is calculated by the following

$$\int \frac{\sqrt{a - e^{-2z}}}{e^{-2z}}dz = -\frac{1}{2\sqrt{a}}arc\tanh\left(\frac{\sqrt{a - e^{-2z}}}{\sqrt{a}}\right) + \frac{\sqrt{a - e^{-2z}}}{2e^{-2z}}.$$

As conclusion, we have

Theorem 2.3. •*The only non extendable flat translation surfaces in Sol_3 which are invariant under the one parameter group of isometries $(x, y, z) \mapsto (x + c, y, z)$, are the surfaces whose parametrization is $X(x, y) = (x, y, z(y))$ where y and z satisfy*

$$y = \pm\left(-\frac{1}{2\sqrt{a}}arc\tanh\left(\frac{\sqrt{a - e^{-2z}}}{\sqrt{a}}\right) + \frac{\sqrt{a - e^{-2z}}}{2e^{-2z}}\right) + \delta,$$

where $a \in \mathbb{R}^{*,+}$, $\delta \in \mathbb{R}$ and $z \in]\ln(\sqrt{a}), +\infty[$.

•*In particular the only complete flat translation surfaces in Sol_3 which are invariant under the one parameter group of isometries $(x, y, z) \mapsto (x + c, y, z)$, are the planes $z = z_0$.*

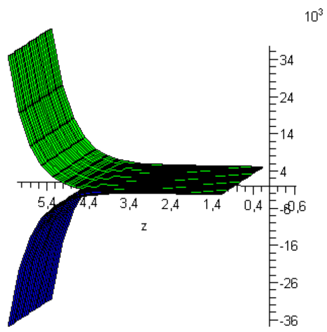

Figure 2: Non extendable flat surface in Sol_3 :$y(z) = \pm\left(\frac{1}{2\sqrt{0.2}}arc\tanh\left(\frac{\sqrt{0.2-e^{-2z}}}{\sqrt{0.2}}\right) - \frac{\sqrt{0.2-e^{-2z}}}{2e^{-2z}}\right) + 2, a = 0.2,\ z = -0.5..6,,\ x = -4..8$.

Theorem 2.4. •*The only complete extrinsically flat translation surfaces in* Sol_3 *which are invariant under the one parameter group of isometries* $(x, y, z) \mapsto (x+c, y, z)$, *are parametrized by*

$$X(x, y) = \left(x, y, \ln\left(\sqrt{-x^2 + 2\lambda x + \mu}\right)\right),$$

where $\lambda, \mu \in \mathbb{R},$ *and* $\lambda^2 + 2\mu > 0\ \alpha \in \mathbb{R}$ *and* $x \in]\lambda - \sqrt{\lambda^2 + 2\mu}, \lambda + \sqrt{\lambda^2 + 2\mu}[$.

Proof. We know that Σ is extrinsically flat surface if and only if $K_{ext} = 0$, and we have $K_{ext} = 0$ equivalent to

$$2z'^2e^{2z}z''e^{2z} = -1.$$

we remark that $2z'^2e^{2z} + z''e^{2z} = (z'e^{2z})'$, thus

$$z'e^{2z} = -x + \lambda, \tag{2.12}$$

where $\lambda \in \mathbb{R}$,and we integrate the equation 2.12

$$z(x) = \ln\left(\sqrt{-x^2 + 2\lambda + 2\mu}\right),$$

where $\mu \in \mathbb{R}$, and $\lambda^2 + 2\mu > 0\ \alpha \in \mathbb{R}$ and $x \in]\lambda - \sqrt{\lambda^2 + 2\mu}, \lambda + \sqrt{\lambda^2 + 2\mu}[$.

□

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ON A TRANSLATED SUM OVER PRIMITIVE SEQUENCES RELATED TO A CONJECTURE OF ERDŐS

NADIR REZZOUG, ILIAS LAIB, AND KENZA GUENDA

Abstract. A strictly increasing sequence $\mathcal{A}$ of positive integers is said to be primitive if no term of $\mathcal{A}$ divides any other. Erdős showed that the series $\sum\limits_{a\in\mathcal{A}} \frac{1}{a\log a}$ for $\mathcal{A}$ different from 1. In this work we show that for x large enough, there exists a primitive sequence $\mathcal{A}$, such that

$$\sum_{a\in\mathcal{A}} \frac{1}{a(\log a + x)} \gg \sum_{p\in\mathcal{P}} \frac{1}{p(\log p + x)},$$

where $\mathcal{P}$ denotes the set of prime numbers.

2010 Mathematics Subject Classification. 11Bxx.

Keywords and phrases. Primitive sequences, Erdős conjecture, Prime numbers.

1. Define the problem

Find an Erdos conjecture proof that $\sum\limits_{a\in\mathcal{A}} \frac{1}{a\log a} \leq \sum\limits_{p\in\mathcal{P}} \frac{1}{p\log p}$, where $\mathcal{A}$ is a primitive sequence and $\mathcal{P}$ is the set of prime numbers.

This problem was stated in 1988.

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ON LATTICE HOMOMORPHISMS IN RIESZ SPACES

ELMILOUD CHIL AND FATEH MEKDOUR

Abstract. In this paper, we study the connection between lattice and Riesz homomorphisms in Riesz spaces. We propose generalized facts of Mena and Roth, Thanh, Lochan and Strauss, and Ercan and Wickstead's approaches (see [3, 6, 8, 11]) for Riesz spaces. To do so, we use new techniques that deal with the prime ideal in Riesz space to prove that any lattice homomorphism in Riesz space is a Riesz homomorphism.

2010 Mathematics Subject Classification. 06F25, 46A40.

Keywords and phrases. Riesz spaces, lattice and Riesz homomorphisms.

1. Define the problem

We begin first from the important historical background on this monograph through the following:

∘ The relation of lattice homomorphisms with Riesz homomorphisms has attracted the attention of many authors in last few decades. The first result in this direction due to Menna and Roth, by their basic works in **1978**. They proved that if X and Y are compact Hausdorff spaces and $T : C(X) \longrightarrow C(Y)$ is a lattice homomorphism such that $T(\lambda 1) = \lambda T(1)$ for all $\lambda \in \mathbb{R}$, then T is linear.

∘ Later, several authors are interested in this problem. Thanh was generalized Mena and Roth's result to the case when X and Y are real compact spaces. For another generalization by Lochan and Strauss.

∘ So far, the best results in this field are duo to Ercan and Wickstead. They showed from the theorem of Mana and Roth by using the Kakutani representation theorem, that if E and F are uniformly complete Archimedean Riesz spaces with weak order units $e_E \in E$ and $e_F \in F$, and if $T : E \longrightarrow F$ is a lattice homomorphism such that $T(\lambda e_E) = \lambda e_F$ for all $\lambda \in \mathbb{R}$, then T is linear.

The motivation for our work appears by the following main ideas:

• Our study is concerned to give the connection between lattice and Riesz homomorphisms in Riesz spaces. We prove under a certain condition that any lattice homomorphism on Riesz space is a Riesz homomorphism.

• Our main goal is to prove that, in results of Ercan and Wickstead, that assumption of the uniform completeness condition on the domain of mapping is superfluous. Down to the general case of Riesz spaces, it seems natural therefore to ask what happens in the general case of Riesz spaces? What about the weaker condition under which a lattice homomorphism defined on a Riesz space is linear?

• As to establish immediate applications of the above result we present a constructive manner for results are obtained more directly constructive.

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ON TERNARY EQUIVALENCE RELATIONS

HAMZA BOUGHAMBOUZ AND LEMNAOUAR ZEDAM

Abstract. In this talk, we introduce the notion of ternary equivalence relations based on the properties of reflexivity, symmetry and transitivity of ternary relations. Due to the various definition of the above properties, we show the appropriate definitions and the undesirable definitions of ternary equivalence relation.

2010 Mathematics Subject Classification. 08A02

Keywords and phrases. Ternary Relation, Relational systems, Equivalence Relations, Equivalence classes.

1. Define the problem

Since the second half of the last century the interest in ternary relations is on the rise [1, 2, 5, 6], driven by the practical application [3, 8]. By far, the most important type of binary relations is the binary equivalence relations. Hence we aim to break through the notion of ternary equivalence relations. which become possible since the recent introduction of the notions of compositions of ternary relations [6], moreover certain compositions has been proven to be associative, which gave rise to the notion of transitive ternary relations[5]. We studied a ternary relation that is (in different senses) reflexive, symmetric and transitive.

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ON THE a-POINTS OF THE k-TH DERIVATIVES OF THE DIRICHLET L-FUNCTIONS

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Abstract. Let $L^{(k)}(s,\chi)$ be the k-th derivative of the Dirichlet L-function associated with a primitive character χ mod q and a be a complex number. The solutions $L^{(k)}(s,\chi)=a$ are called a-points. In this talk, we present our results [3] for the sums

$$\sum_{\rho_{0,\chi}^{(k)}:0<\gamma_{0,\chi}^{(k)}<T} L^{(j)}(\rho_{0,\chi}^{(k)},\chi) \textit{ and } \sum_{\rho_{a,\chi}^{(k)}:1<\gamma_{a,\chi}^{(k)}<T} L^{(j)}(\rho_{a,\chi}^{(k)},\chi) \textit{ as } T\to\infty$$

where j and k are non-negative integers and $\rho_{a,\chi}^{(k)}$ denotes an a-point of the k-th derivative $L^{(k)}(s,\chi)$ and $\gamma_{a,\chi}^{(k)}=\mathrm{Im}(\rho_{a,\chi}^{(k)})$. This work continues the investigations of Kaptan, Karabulut & Yildirim [1, 2] and Mazhouda & Onozuka [4].

2010 Mathematics Subject Classification. 11M06,11M26, 11M36.

Keywords and phrases. Dirichlet L-function, a-points, value-distribution.

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On the compactness in standard single valued neutrosophic metric spaces

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Abstract

Recently, we have introduced the notion of standard single valued neutrosophic metric space as a generalization of standard fuzzy metric space given by J.R. Kider and Z.A. Hussain, where some interesting properties have been investigated such as the continuity property of the mappings defined on standard single valued neutrosophic metric spaces.

In this work, we continue our previous study by introducing the notion of compact standard single valued neutrosophic metric space. Moreover, we give a certain number of properties and characterizations of this notion and the relationships between them.

Keywords: Metric space, Single valued neutrosophic set, Compactness.

ON THE INTERSECTION OF k-LUCAS SEQUENCES AND SOME BINARY SEQUENCES

SALAH EDDINE RIHANE AND ALAIN TOGBÉ

Abstract. For an integer $k \geq 2$, let $(L_n^{(k)})_n$ be the k-generalized Lucas sequence which starts with $0, \ldots, 0, 2, 1$ (k terms) and each term afterwards is the sum of the k preceding terms. In this paper, we find all the k-generalized Lucas numbers which are Fibonacci, Pell or Pell-Lucas numbers i.e., we study the Diophantine equations $L_n^{(k)} = F_m$, $L_n^{(k)} = P_m$ and $L_n^{(k)} = Q_m$ in positive integers n, m, k with $k \geq 3$.

2010 Mathematics Subject Classification. 11B39, 11J86.

Keywords and phrases. k-generalized Lucas numbers, Fibonacci numbers, Pell numbers, Pell-Lucas numbers, Linear form in logarithms, reduction method.

1. Introduction

The Fibonacci $(F_n)_{n\geq 0}$ and Lucas $(L_n)_{n\geq 0}$ sequences are given by

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad \text{for all } n \geq 2$$

and

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2}, \quad \text{for all } n \geq 2,$$

receptively. A few terms of these sequences are

$$(F_n)_{n\geq 0} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \cdots\}$$

and

$$(L_n)_{n\geq 0} = \{2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, \cdots\}.$$

The Pell $(P_n)_{n\geq 0}$ and Pell-Lucas $(Q_n)_{n\geq 0}$ sequences are given by

$$P_0 = 0, \quad P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2}, \quad \text{for all } n \geq 2$$

and

$$Q_0 = 2, \quad Q_1 = 2, \quad Q_n = 2Q_{n-1} + Q_{n-2}, \quad \text{for all } n \geq 2,$$

receptively. A few terms of these sequences are

$$(P_n)_{n\geq 0} = \{0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, \cdots\}$$

and

$$(Q_n)_{n\geq 0} = \{2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, 16238, 39202, 94642, \cdots\}.$$

In [1], Alekseyev prove that $F \cap L = \{1, 2, 3\}$ and $P \cap L = \{1, 2, 29\}$.

Let $k \geq 2$ be an integer. We consider a generalization of Lucas sequence called the k-generalized Lucas sequence $L_n^{(k)}$ defined as

$$L_n^{(k)} = L_{n-1}^{(k)} + L_{n-2}^{(k)} + \cdots + L_{n-k}^{(k)}, \quad \text{for all } n \geq 2, \tag{1}$$

with the initial conditions $L^{(k)}_{-(k-2)} = L^{(k)}_{-(k-3)} = \cdots L^{(k)}_{-1} = 0$, $L^{(k)}_0 = 2$, and $L^{(k)}_1 = 1$. If $k = 2$, we obtain the classical Lucas sequence *i.e* $L^{(2)}_n = L_n$. If $k = 3$, then the 3-Lucas sequence is

$(L^{(3)}_n)_{n\geq -1} = \{0, 2, 1, 3, 6, 10, 19, 35, 64, 118, 217, 399, 734, 1350, 2483, 4567, \ldots\}.$

If $k = 4$, then the 4-Lucas sequence is

$(L^{(4)}_n)_{n\geq -2} = \{0, 0, 2, 1, 3, 6, 12, 22, 43, 83, 160, 308, 594, 1145, 2207, 4254, 8200, \ldots\}.$

2. Main results

In [16], we extend the result of Alekseyev, more precisely, we solve the Diophantine equations

$$L^{(k)}_n = F_m, \tag{2}$$

$$L^{(k)}_n = P_m \tag{3}$$

and

$$L^{(k)}_n = Q_m. \tag{4}$$

We show the following results.

Theorem 2.1. *All the integer solutions* (n, m, k) *of Diophantine equation* (2) *are*

$$(0, 3, k), \quad (1, 1, k), \quad (1, 2, k) \quad \textit{and} \quad (2, 4, k).$$

Thus $F \cap L^{(k)} = \{1, 2, 3\}$.

Theorem 2.2. *All the integer solutions* (n, m, k) *of Diophantine equation* (3) *with* $k \geq 4$ *are*

$$(0, 2, k), \quad (1, 1, k) \quad \textit{and} \quad (4, 4, k).$$

If $k = 3$, *then all the integer solutions* (n, m, k) *of Diophantine equation* (3) *are*

$$(0, 2, 3) \quad \textit{and} \quad (1, 1, 3).$$

Hence, $P \cap L^{(k)} = \{1, 2, 12\}$.

Theorem 2.3. *All the integer solutions* (n, m, k) *of Diophantine equation* (4) *with* $k \geq 3$ *are*

$$(0, 0, k), \quad (0, 1, k) \quad \textit{and} \quad (3, 2, k).$$

Therefore, $Q \cap L^{(k)} = \{2, 6\}$.

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PRESCRIBED Q-CURVATURE TYPE PROBLEM ON COMPACT MANIFOLDS

MOHAMED BEKIRI

ABSTRACT. In this work, we investigate the existence of changing-sign solution to Dirichlet elliptic problem involving Paneitz-Branson type operator on compact Riemannian manifold with boundary.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 53A30, 58J05, 53C21.

KEYWORDS AND PHRASES. Dirichlet problem, Paneitz-Branson type operator, Sobolev critical exponent.

1. DEFINE THE PROBLEM

Given (M, g) be a smooth Riemannian compact manifold with boundary of dimension $(n \geq 5)$. We let A be a smooth symmetric $(2, 0)$-tensor on M and $a \in C^{\infty}(M)$.
The goal in this work is to study the following Dirichlet elliptic problem

$$\begin{cases} P_g u = \lambda f \, |u|^{2^{\sharp}-2} u & \text{in } M \\ u = \phi_1, \, \partial_\nu u = \phi_2 & \text{on } \partial M \end{cases} \tag{1}$$

where

$$P_g u = \Delta_g^2 u - \operatorname{div}_g \left(A \, (\nabla u)^{\#} \right) + au. \tag{2}$$

is the Paneitz-Branson type operator, $\phi_1, \phi_2 \in C^{\infty}(\partial M)$ are boundary data such as ϕ_1 is a sign-changing function, $f \in C^{\infty}(M)$ is a positive function and $2^{\sharp} = \frac{2n}{n-4}$ is the Sobolev critical exponent.
More precisely, we want to find some conditions on the operator P_g and f, for the equation (1) to have a nodal (changing-sign) solution $u \in H^2_{2,0}(M) \cap C^{4,\alpha}(M)$.
The problem (1) has the peculiarity of containing the critical Sobolev exponent, which leads us to use the variational approach developed by Yamabe [4] and used by Holcman [3].
The problem (1) is equivalent to the following problem

$$\begin{cases} P_g w = \lambda f \, |w + h|^{2^{\sharp}-2} (w + h) & \text{in } M \\ w = \partial_\nu w = 0 & \text{on } \partial M \end{cases} \tag{3}$$

where $h \in H_2^2(M) \cap C^{4,\alpha}(M)$ is the unique solution of the following linear problem

$$\begin{cases} \Delta_g^2 h - \operatorname{div}_g \left(A \, (\nabla h)^{\#} \right) + ah = 0 & \text{in } M \\ h = \phi_1 \text{ and } \partial_\nu h = \phi_2 & \text{on } \partial M \end{cases}$$

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PROPERTIES OF HOMODERIVATIONS ON LATTICE STRUCTURES

MOURAD YETTOU AND ABDELAZIZ AMROUNE

Abstract. In this paper, the concept of homoderivation on a lattice as a combination of two concepts of *meet*-homomorphisms and derivations is introduced. Some characterizations and properties of homoderivations are provided. The relationship between derivations and homoderivations on a lattice is established. Also, an interesting class of homoderivations namely isotone homoderivations is studied. A characterization of the isotone homoderivations in terms of the *meet*-homomorphisms is given. Furthermore, a sufficient condition for a homoderivation to become isotonic is established.

2010 Mathematics Subject Classification. 03G10, 06B05, 06B10, 06B99.

Keywords and phrases. Lattice, derivation, homoderivation, isotone homoderivation.

1. Define the problem

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RICCI-PSEUDO-SYMMETRIC GENERALIZED S-SPACE-FORMS

RACHIDA KAID AND MOHAMED BELKHELFA

Abstract. The purpose of this work is to examine the problem of symmetry properties of the generalized S-space-form with two structure vectors fields, that generalized naturally the S-space-form $M^{2n+s}(c)$ which is not pseudo-symmetric for $s \geq 2, n \geq 1$ and $c \neq s$. However for $s = 1$ this one is reduced to Sasakian space-form, which is pseudo-symmetric. We establish that, under some conditions, particular generalized S-spaces form can be Ricci pseudo-symmetric.

2010 Mathematics Subject Classification. 53A55, 53B20, 53C35.

Keywords and phrases. generalized Sasakian space form, generalized S-space-form, Ricci-pseudo-symmetry, .

1. Define the problem

K. Yano [8] introduced the notion of f-structure on a $(2n+s)$- dimensional manifold as a tensor field f of type $(1,1)$ and rank $2n$ satisfying $f^3 + f = 0$. Almost complex and almost contact structures, for respectively $s = 0$ ans $s = 1$, are well-known examples of f-structures. D. E. Blair [3] introduced the K-structure on a manifold M^{2n+s} with an f-structure, as the analogue of the Kähler structure in the almost complex case and of the quasi-Sasakian structure in the almost contact case. The S-manifold is a class of the K-manifold and its curvature tensor is completely determined by the f-sectional curvature. When the f-sectional curvature is constant, the S-manifold is said to be a S-space form. Later, M. Kobayashi and S. Tsuchiya in [7] got expression of the curvature tensor field of a S-space form. In [1] is introduced the notion of a generalized Sasakian space form as an almost contact metric manifold (M, f, ξ, η, g) whose curvature tensor satisfies

$$\begin{aligned} R(X,Y)Z &= f_1(g(Y,Z)X - g(X,Z)Y) \\ &+ f_2(g(X,fZ)fY - g(Y,fZ)fX \\ &+ 2g(X,fY)fZ) \\ &+ f_3(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi) \end{aligned} \tag{1}$$

for all vector fields X, Y, Z and certain differentiable functions f_1, f_2, f_3 on M. This generalizes the concept of Sasakian space form as well as generalized complex space form did with complex space form. If $f_1 = \dfrac{c+3}{4}$ and $f_2 =$

$f_3 = \dfrac{c-1}{4}$, the generalized Sasakian-space-form is reduced to a Sasakian-space-form with f-sectional curvature c.

The generalized S-space-form with two structure vectors fields [4], generalized naturally the S-space-form $M^{2n+s}(c)$ where c is the f-sectionnal curvature, in the same way as generalized Sasakian space forms generalized the Sasakian space forms, and it is defined as a metric f-manifold $(M, f, \eta_1, \eta_2, \xi_1, \xi_2, g)$ with two structure vector fields ξ_1 and ξ_2 such that the curvature tensor field satisfies (see [4])

$$
\begin{aligned}
\mathrm{R}(X,Y)Z = {} & F_1\{g(Y,Z)X - g(X,Z)Y\} \\
& + F_2\{g(X,fZ)fY - g(Y,fZ)fX + 2g(X,fY)fZ\} \\
& + F_3\{\eta_1(X)\eta_1(Z)Y - \eta_1(Y)\eta_1(Z)X + g(X,Z)\eta_1(Y)\xi_1 \\
& - g(Y,Z)\eta_1(X)\xi_1\} \\
& + F_4\{\eta_2(X)\eta_2(Z)Y - \eta_2(Y)\eta_2(Z)X + g(X,Z)\eta_2(Y)\xi_2 \\
& - g(Y,Z)\eta_2(X)\xi_2\} \\
& + F_5\{\eta_1(X)\eta_2(Z)Y - \eta_1(Y)\eta_2(Z)X + g(X,Z)\eta_1(Y)\xi_2 \\
& - g(Y,Z)\eta_1(X)\xi_2\} \\
& + F_6\{\eta_2(X)\eta_1(Z)Y - \eta_2(Y)\eta_1(Z)X + g(X,Z)\eta_2(Y)\xi_1 \\
& - g(Y,Z)\eta_2(X)\xi_1\} \\
& + F_7\{\eta_1(X)\eta_2(Y)\eta_2(Z)\xi_1 - \eta_2(X)\eta_1(Y)\eta_2(Z)\xi_1\} \\
& + F_8\{\eta_2(X)\eta_1(Y)\eta_1(Z)\xi_2 - \eta_1(X)\eta_2(Y)\eta_1(Z)\xi_2\}
\end{aligned} \tag{2}
$$

for any $X, Y, Z \in \chi(M)$ and where $F_1, \ldots, F_8$ are differentiable functions on M.

The aim of this work, is to look at the problem of symmetry properties of the generalized S-space-form with two structure vectors fields. The S-space-forms $M^{2n+s}(c)$, are not pseudo-symmetric for $s \geq 2, n \geq 1$ and $c \neq s$ [6]. For $s = 1$ this one is reduced to Sasakian space-form, in this case the second author and al [2] have shown that it is pseudo-symmetric. We study the Ricci pseudo-symmetry for the generalized S-space-forms. We establish that the generalized S-space-forms with two structure vectors fields which are metric f-K-contact manifolds and then S-manifolds, can't be Ricci pseudo-symmetric, we also studied the generalized S-space-forms which are the warped product $M = \mathbb{R} \times_h \widetilde{M}$, where $h > 0$ is a differentiable function on $\mathbb{R}$ and $\widetilde{M} = \widetilde{M}(f_1, f_2, f_3)$ [4] is a genelized Sasakian space form. We give conditions on the functions h and f_1, f_2, f_3 under which these generalized S-spaces forms can be Ricci pseudo-symmetric.

We have organized the paper in the following way: we introduce the subject and his context in the first section, in Section 2, we recall the notions of Ricci pseudo-symmetry [5] and we review basic formulas of metric f-manifolds meaning. The third Section, is devoted to the generalized S-space-forms. Finally, in Section 4 we look at the problem of symmetry properties of the generalized S-space-form with two structure vectors fields [4].

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REPRESENTATION OF THE FUZZY RELATIONS THAT A BINARY RELATION IS COMPATIBLE WITH.

HASSANE BOUREMEL

Abstract. The notion of compatibility expresses that elements that are similarly related to other related elements are related as well. This notion is an important extension of the extensionality of a mapping between two universes with L-fuzzy equality was introduced by Höhle and Blanchard.
Also this notion is a similar of the compatibility of a fuzzy relation with respect to a L-fuzzy equality and L-fuzzy equivalence rlations was introduced by Bělohlávek on the fuzzy approach to concept lattices.

The main aim of this work is the representation of some types of the L-fuzzy relations that a binary relation is compatible with.

Keywords and phrases. Lattice, residuated lattice, fuzzy set, fuzzy relation, Clone relation, Compatibility.

1. Define the problem

Solving the general problem of characterizing the fuzzy equivalence relations a given binary relation is compatible with by mean of the clone relation.

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SOME BIHARMONIC PROBLEMS ON THE TANGENT BUNDLE WITH A BERGER-TYPE DEFORMED SASAKI METRIC

ABDALLAH MEDJADJ, HICHEM EL HENDI, AND BOUAZZA KACIMI

ABSTRACT. Let (M_{2k}, ϕ, g) be an almost anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. In this paper, we deal with the biharmonicity of canonical projection $\pi : TM \longrightarrow M$ and a vector field X which is considered as a map $X : M \longrightarrow TM$.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 53C07, 53C15.

KEYWORDS AND PHRASES. Berger type deformed Sasaki metric, anti-paraKähler manifold, biharmonic map.

1. Define the problem

Studying the harmonocity and the biharmonicity of map in an almost anti-paraKähler manifold and TM its tangent bundle.

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Title: Some identities of Mersenne-Lucas numbers and generationg function of their products

Mourad Chelgham and Ali Boussayoud.

Abstract

In this work, we will introduce new definition of k-Mersenne-Lucas numbers and investigate some properties. Then, we obtain some identities and estabished connection formulas between k-Mersenne-Lucas numbers and k-Mersenne numbers through use of Binet's formula. We also give the generating function of their Hadamard products (square, successive terms and non successive terms) using symetric functions technic.

SOME PROPERTIES OF TERNARY RELATIONS AND THEIR CLOSURES

NORELHOUDA BAKRI AND LEMNAOUAR ZEDAM

Abstract. We study the problem of closing a ternary relation with respect to various relational properties, for instance, reflexivity, symmetry and cyclicity with a focus on the many transitivity properties that have been proposed for ternary relations over the past years.

2010 Mathematics Subject Classification. 03E20, 97E60.

Keywords and phrases. Ternary relation, transitivity, closure.

1. Define the problem

Relations come in many flavors, such as binary or ternary, crisp or fuzzy, et cetera. Although generally less popular than binary relations, ternary relations also play a diverse role in many branches of mathematics. In recent years, the interest in ternary relations is on the rise [2, 3]. Ternary relations can display various interesting properties, such as reflexivity, symmetry, cyclicity and transitivity, some of which do not exist in the binary case (such as cyclicity) or come in a multitude of variations in the ternary case (such as transitivity). In case a binary relation R does not possess a desired property P, the question arises whether it is possible to find (if it exists) the smallest binary relation including R and possessing property P, which is called its P-closure. The main aim is to derive results similar to those of Bandler and Kohout [1] for the setting of ternary relations.

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SOME PROPERTIES OF TRELLISES

ABDELKRIM MEHENNI AND LEMNAOUAR ZEDAM

Abstract. In the present paper, we study an extended structure of a lattice (trellis, for short) by considering sets with a reflexive and antisymmetric, but not necessarily transitive relation. Of course, by postulating the existence of least upper bounds and greatest lower bounds of each pair of elements similarly to the case of lattices. Also, we present some properties analogous to nearly all the basic theorems of lattice theory, thus demonstrating the superfluity of the assumption of associativity.

2010 Mathematics Subject Classification. 06B05, 06B15.

Keywords and phrases. Psoset, trellis, binary operation, associative element, transitive element.

1. Define the problem

The material presented herein is a generalization of the concepts of partial order and lattice [3, 4, 8, 10]. By starting out with a reflexive and antisymmetric, but not necessarily transitive, order, we define least upper bound and greatest lower bound similarly as for partially ordered sets, thus obtaining a structure, called a trellis [5, 11], in which these operations are not necessarily associative. With this approach we can prove nearly all the basic theorems of lattice theory, thus demonstrating the superfluity of the assumption of associativity. Moreover, in the presence of certain additional assumptions, associativity follows as a consequence.

The ideas of transitivity and partial order are, without question, fundamental in a wide variety of mathematical theories. The mathematical underground, however, has been simmering for some time with notions of non-transitive relationssome arising from common, every-day observations and some from purely mathematical considerations.

An important step in the theory of partial orderings was the postulation of least upper bounds and greatest lower bounds and the development of the theory of lattices. Transitivity is necessary for the associativity of the operations of least upper bound and greatest lower bound. And associativity has been regarded as essential to the theory of lattices as the proofs of many theorems heavily depend upon it. So it would seem that transitivity is an indispensable requirement for lattice theory. However, starting out with a reflexive and antisymmetric, but not necessarily transitive, order, we can define least upper bounds and greatest lower bounds similarly as for partially ordered sets. With this approach we can prove theorems analogous to nearly all the basic theorems of lattice theory, thus demonstrating the superfluity of the assumption of associativity. Moreover, in the presence of certain additional assumptions, such as distributivity, relative complementation and modularity, or others, associativity follows as a consequence.

The material herein presented contains a foundation for the theory of non-transitive orderings. By stipulation of the existence of least upper bounds and greatest lower bounds we obtain a structure, called a trellis, having properties similar to those of lattices. It is indeed surprising how much can be done under so few assumptions.

In the present paper we study a generalization of lattices by considering sets with a reflexive and antisymmetric, but not necessarily transitive, relation and by postulating the existence of least upper bounds and greatest lower bounds similarly as for partially ordered sets; and, alternatively, by considering sets with two operations that are commutative, absorptive, and, what will be called, part-preserving. Using this approach we are able to prove theorems analogous to nearly all the basic theorems of lattice theory, thus demonstrating the superfluity of the assumption of associativity. Moreover, in the presence of certain additional assumptions, such as distributivity, relative complementation and modularity, or others, associativity follows as a consequence.

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Title : **Surfaces of finite type in $\widetilde{SL}(2,\mathbb{R})$**

Ahmed Azzi, Zoubir Hanifi and Mohammed Bekkar

Abstact : In this paper, we prove that $\Delta X = 2H$ where Δ is Laplacian operator, $X(r,\theta,\phi)$ the position vector field and H is the mean curvature vector field of surface S in $\widetilde{SL}(2,\mathbb{R})$ and we study surfaces as graph in $\widetilde{SL}(2,\mathbb{R})$ which has finite type immersion.

Mathematics Subject Classification: 53B05; 53B21; 53C30.

Keywords: Laplacian operator, $\widetilde{SL}(2,\mathbb{R})$ geometry, surfaces of coordinate finite type.

In this work we study the surfaces as graphs of functions $\phi = f(r,\theta)$ in $\widetilde{SL}(2,\mathbb{R})$ satisfy the condition:

$$\Delta x_i = \lambda_i x_i$$

where $\lambda_i \in \mathbb{R}$ and x_i are the coordinate functions of the surface.

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THE SKEW REVERSIBLE CODES OVER FINITE FIELDS

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Abstract. In this paper we give a necessary and sufficient condition for a skew λ-constacyclic code generated by a skew polynomial $g(x)$ (not necessarily central) to be a LCD code under some assumptions. We make some link with skew reversible codes and conjugate-skew reversible codes.

Keywords and phrases. Skew polynomial rings, Skew constacyclic codes, LCD codes, conjugate-skew reversible codes, skew reversible codes.

1. Preliminaries

Let q be a prime power, $\mathbb{F}_q$ a finite field and θ an automorphism of $\mathbb{F}_q$. We define the skew polynomial ring R as

$$R = \mathbb{F}_q[x;\theta] = \{a_0 + a_1x + \ldots + a_{n-1}x^{n-1} \mid a_i \in \mathbb{F}_q \text{ and } n \in \mathbb{N}\}$$

under usual addition of polynomials and where multiplication is defined using the rule

$$\forall a \in \mathbb{F}_q, x \cdot a = \theta(a)x.$$

The ring R is noncommutative unless θ is the identity automorphism on $\mathbb{F}_q$. According to [9], an element f in R is central if and only if f is in $\mathbb{F}_q^\theta[x^\mu]$ where μ is the order of the automorphism θ and $\mathbb{F}_q^\theta$ is the fixed field of θ. The two-sided ideals of R are generated by elements having the form $(c_0+c_1x^\mu+\ldots+c_nx^{n\mu})x^l$, where l is an integer and c_i belongs to $\mathbb{F}_q^\theta$. Central elements of R are the generators of two-sided ideals in R [3]. The ring R is Euclidean on the right : the division on the right is defined as follows. Let f and g be in R with $f \neq 0$. Then there exist unique skew polynomials q and r such that

$$g = q \cdot f + r \text{ and } \deg(r) < \deg(f).$$

If $r = 0$ then f is a right divisor of g in R ([9]). There exist greatest common right divisors (gcrd) and least common left multiples (lclm). The ring R is also Euclidean on the left : there exist a division on the left, greatest common left divisors (gcld) as well as least common right multiples (lcrm).

In what follows, we consider a positive integer n and a constant λ in $\mathbb{F}_q^*$.

According to [3] and [5], a linear code C of length n over $\mathbb{F}_q$ is said to be (θ,λ)**-constacyclic** or **skew λ-constacyclic** if it satisfies

$$\forall c \in \mathbb{F}_q^n, c = (c_0, c_1, \ldots, c_{n-1}) \in C \Rightarrow (\lambda\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in C.$$

Any element of the left R-module $R/R(x^n-\lambda)$ is uniquely represented by a polynomial $c_0 + c_1x + \ldots + c_{n-1}x^{n-1}$ of degree less than n, hence is identified with a word $(c_0, c_1, \ldots, c_{n-1})$ of length n over $\mathbb{F}_q$.

In this way, any skew λ-constacyclic code C of length n over $\mathbb{F}_q$ is identified with exactly one left R-submodule of the left R-module $R/R(x^n-\lambda)$, which is generated by a right divisor g of $x^n-\lambda$. In that case, g is called a **skew generator polynomial** of C and we will denote $C=\langle g\rangle_n$.

Note that the skew 1-constacyclic codes are skew cyclic codes and the skew (-1)-constacyclic codes are skew negacyclic codes.

The **Hamming weight** $wt(y)$ of an n-tuple $y=(y_l,y_2,\ldots,y_n)$ in $\mathbb{F}_q^n$ is the number of nonzero entries in y, that is, $wt(y)=\mid\{i: y_i\neq 0\}\mid$. The **minimum distance** of a linear code C is $\min_{c\in C, c\neq 0} wt(c)$.

The **Euclidean dual** of a linear code C of length n over $\mathbb{F}_q$ is defined as $C^\perp=\{x\in\mathbb{F}_q^n \mid \forall y\in C, <x,y>=0\}$ where for x,y in $\mathbb{F}_q^n$, $<x,y>:=\sum_{i=1}^n x_iy_i$ is the (Euclidean) scalar product of x and y. A linear code is called an **Eulidean LCD** code if $C\oplus C^\perp=\mathbb{F}_q^n$, which is equivalent to $C\cap C^\perp=\{0\}$.

Assume that $q=r^2$ is an even power of an arbitrary prime and denote for a in $\mathbb{F}_q$, $\overline{a}=a^r$. The **Hermitian dual** of a linear code C of length n over $\mathbb{F}_q$ is defined as $C^{\perp_H}=\{x\in\mathbb{F}_q^n \mid \forall y\in C, <x,y>_H=0\}$ where for x,y in $\mathbb{F}_q^n$, $<x,y>_H:=\sum_{i=1}^n x_i\overline{y_i}$ is the (Hermitian) scalar product of x and y. The code C is a **Hermitian LCD** code if $C\cap C^{\perp_H}=\{0\}$.

The **skew reciprocal polynomial** of $g=\Sigma_{i=0}^k g_ix^i\in R$ of degree k is $g^*=\Sigma_{i=0}^k\theta^i(g_{k-i})x^i$. If g_0 does not cancel, the left monic skew reciprocal polynomial of g is $g^\natural=(1/\theta^k(g_0))g^*$. If a skew polynomial is equal to its left monic skew reciprocal polynomial, then it is called self-reciprocal.

Consider C a skew λ-constacyclic code of length n and skew generator polynomial g. According to Theorem 1 and Lemma 2 of [4], the Euclidean dual $C^\perp$ of C is a skew λ^{-1}-constacyclic code generated by $h^\natural$ where $\Theta^n(h)\cdot g=x^n-\lambda$ and for $a(x)=\sum a_ix^i\in R$, $\Theta(a(x)):=\sum\theta(a_i)x^i$. In particular, when λ is fixed by θ and n is a multiple of the order μ of θ, then h is fixed by Θ^n and $x^n-\lambda$ is central, therefore one gets $h\cdot g=g\cdot h=x^n-\lambda$. If $q=r^2$, the Hermitian dual $C^{\perp_H}$ of C is generated by $\overline{h^\natural}$ where for $a(x)=\sum a_ix^i\in R$, $\overline{a(x)}:=\sum\overline{a_i}x^i$.

Lemma 1.1. [2, Lemma 4] *Consider h and g in R. Then $(h\cdot g)^*=\Theta^{\deg(h)}(g^*)\cdot h^*$.*

In the following, we give a necessary and sufficient condition for a skew λ-constacyclic code to be a LCD code when $\lambda^2=1$.

Theorem 1.2. [10, Theorem 4.1] *Assume that $\lambda^2=1$. Consider a skew λ-constacyclic code C with skew generator polynomial g and length n. Consider h such that $hg=gh=x^n-\lambda$.*

(1) *C is an Euclidean LCD if and only if* $\mathrm{gcrd}(g,h^\natural)=1$.
(2) *If q is an even power of a prime number, $q=r^2$, C is an Hermitian LCD code if and only if* $\mathrm{gcrd}(g,\overline{h^\natural})=1$.

1.1. **LCD and skew reversible skew constacyclic codes.** Over a finite field $\mathbb{F}_q$, there is a strong link between LCD cyclic codes and reversible codes ([6], [8]).

Definition 1.3. *Let $\mathbb{F}_q$ be the finite field where q is a prime power. A code C is called **reversible** if $(c_0, c_1, \ldots, c_{n-1}) \in C$ implies that $(c_{n-1}, c_{n-2}, \ldots, c_0) \in C$.*

The cyclic code generated by the monic polynomial g is reversible if and only if $g(x)$ is self-reciprocal (i.e $g(x) = g^\sharp(x)$) [8, Theorem 1]. Furthermore, if q is coprime with n, a cyclic code of length n is LCD if and only if C is reversible [6]. The example below shows that it is not necessarily the case for skew cyclic codes when θ is not the identity.

Example 1.4. *Let $\mathbb{F}_9 = \mathbb{F}_3(w)$ where $w^2 = w + 1$, θ the Frobenius automorphism and $R = \mathbb{F}_9[x;\theta]$. We have :*

$$x^2 - 1 = (x + w^2) \cdot (x + w^2)\cdot$$

The skew polynomial $g = x + w^2$ is such that $g(x) = g^\sharp(x)$. The greatest common right divisor of $g(x)$ and $h^*(x)$ is $x+w^2$ (i.e $gcrd(g(x), h^*(x)) \neq 1$) therefore, by Theorem 1.2, C is not an LCD code.

Definition 1.5. *Let $\mathbb{F}_q$ be the finite field where q is a prime power and θ be an automorphism of $\mathbb{F}_q$, C be a code of length n over $\mathbb{F}_q$.*

(1) *The code C is called a **skew reversible** code if*

$$\forall c \in C \;\; c = (c_0, c_1, \ldots, c_{n-1}) \in C \Longrightarrow \big(c_{n-1}, \theta(c_{n-2}), \ldots, \theta^{n-1}(c_0)\big) \in C$$

(2) *If q is an even power, $q = r^2$, C is a **conjugate-skew reversible** code if*

$$\forall c \in C \;\; c = (c_0, c_1, \ldots, c_{n-1}) \in C \Longrightarrow \big(\overline{c_{n-1}}, \theta(\overline{c_{n-2}}), \ldots, \theta^{n-1}(\overline{c_0})\big) \in C$$

Before we prove our main results in this section, we define the following set: let f, g in R such that $gcrd(f(x), g(x)) = 1$,

$$A_{(f,g)} := \big\{(a(x), b(x)) \in R^2 \mid a(x)f(x) + b(x)g(x) = 1 \text{ and } b(x)g(x) = g(x)b(x)\big\}.$$

Theorem 1.6. *Consider g, h in R and $\lambda \in \{-1, 1\}$ such that $x^n - \lambda = g \cdot h = h \cdot g$ with $\deg(h) = k$.*

(1) *Assume that $A_{(g,\Theta^b(h^*))}$ is nonempty. Then $g = \Theta^{k+b}(g^\natural)$ for all b in $\{0, 1\}$.*
(2) *If the greatest common right divisor of $h(x)$ and $g(x)$ is equal to 1, g_0 in $\mathbb{F}_q^\theta$ and $g = \Theta^{k+b}(g^\natural)$ then $gcrd(g(x), \Theta^b(h^\natural(x)))) = 1$ for all b in $\{0, 1\}$.*
(3) *If the greatest common left divisor of g and h is equal to 1 and if $g = \Theta^b(g^\natural)$, then $gcrd(g(x), \Theta^b(h^\natural(x))) = 1$ for all b in $\{0, 1\}$.*

In the following, using the Theorem 1.6 we give a necessary and sufficient condition for a skew constacyclic code to be an Euclidean LCD code and Hermitian LCD code.

Corollary 1.7. *Consider C a skew λ-constacyclic code with skew generator g and length n.*

(1) *If $A_{(g,h^*)}$ is nonempty and if C is an Euclidean LCD skew constacyclic code then $g = \Theta^k(g^\natural)$.*
(2) *If $A_{(g,\Theta(h^*))}$ is nonempty and if C is an Hermitian LCD skew constacyclic code then $g = \Theta^{k+1}(g^\natural)$.*

(3) *If the greatest common right divisor of $h(x)$ and $g(x)$ is equal to 1, g_0 in $\mathbb{F}_q^\theta$ and $g(x) = \Theta^{k+b}(g^\natural(x))$ then C is an Euclidean LCD skew constacyclic code when $b = 0$ and C is an Hermitian LCD skew constacyclic code when $b = 1$.*

(4) *If the greatest common left divisor of $h(x)$ and $g(x)$ is equal to 1 and $g = \Theta^b(g^\natural)$ then C is an Euclidean LCD skew constacyclic code when $b = 0$ and C is an Hermitian LCD skew constacyclic code when $b = 1$.*

(5) *If the greatest common left divisor of $h(x)$ and $g(x)$ is equal to 1 and C is a skew reversible code (resp.conjugate-skew reversible code) then C is an Euclidean LCD skew constacyclic code (resp. C is an Hermitian LCD skew constacyclic code).*

Remark 1.8. *Suppose $gcrd(g(x), \Theta^b(h^*))) = 1$, there are polynomials $(a(x), b(x)) \in A_{(g,\Theta^b(h^*))}$, as a special case of Corollary 1.7 we obtain C is an Euclidean LCD skew constacyclic code or C is an Hermitian LCD skew constacyclic code then $g = \Theta^b(g^\natural)$ when the order μ of θ divides k.*

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TRIVECTORS OF RANK 8 OVER A FINITE FIELD

NOUREDDINE MIDOUNE

ABSTRACT
For vector spaces of dimension 8 over a finite field $\mathbf{F}_q$ of characteristic 2 all trilinear alternating forms are determined.

Mathematics Subject Classification : Primary 15A69, Secondary 15A75, 05A15, 15A18
Keywords **:** trivector, isotropy groups, classification.

DEFINE THE PROBLEM

Let V be an 8-dimensional vector space over a field K and let $\wedge^3 V$ denote the exterior power of degree 3 over K. The classification of trivectors is the study of the action of general linear group $GL(V)$ on the K-vector space $\wedge^3 V$. By virtue of the canonical identification $\wedge^3 V^* \simeq (\wedge^3 V)^*$, $f.\omega = (\wedge^3 f)(\omega)$, there is no difference between trivectors and trilinear alternating forms. This classification is motivated by many important applications, especially in the theory of codes ([8]) and generalized elliptic curves [1]. G.B.Gurevitch [3], D. Djokovic [2], L.Noui [7], N.Midoune and L.Noui [5] , J.Hora and P.Pudlak [4] give an answer to the classification with $K = C, K = R$, K algebraically closed field of arbitrary characteristic, K a finite field of characteristic other than 2 and 3 and $K = \mathbb{Z}/2\mathbb{Z}$ respectively.

We are interested in classification of trivectors of an eight dimensional vector space over a finite field of characteristic 2. By this result we have, in particular, for $q = 2$ the theorem of J.Hora[4] .

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THE AVERAGE HULL DIMENSION OF CYCLIC CODES OVER FINITE NON CHAIN RINGS

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ABSTRACT. In this work, we study the hull of cyclic codes over a finite non chain ring $\mathcal{R} = \mathbb{F}_q + v\mathbb{F}_q$, where $v^2 = v$. The main tool for the characterization of the hull of cyclic codes is given in term of their generator polynomials. We also establish the average $q-$dimension of the hull of cyclic codes of length n over $\mathcal{R}$. The formula for the average $q-$dimension of the hull of cyclic codes of length n over $\mathcal{R}$ is derived.

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KEYWORDS AND PHRASES. Hulls, Cyclic codes, Average $q-$dimension.

1. PRELIMINARY

Let $\mathcal{R} = \mathbb{F}_q + v\mathbb{F}_q$, where $q = p^m$, p is prime number and $v^2 = v$. Clearly $\mathcal{R} \simeq \mathbb{F}_q[v]/ < v^2 - v >$ is a non chain ring with q^2 element. It is a semi-local ring with maximal ideals $< v >$ and $< 1 - v >$. $\mathcal{R}$ is commutative ring with identity and characteristic p.

A linear code C of length n over $\mathcal{R}$ is defined to be an $\mathcal{R}-$submodule of $\mathcal{R}^n$. the Euclidean dual of C is defined to be the set

$$C^{\perp} = \left\{ (x_0, x_1, ..., x_{n-1}) \in \mathcal{R}^n | \sum_{i=0}^{n-1} x_i c_i = 0 \text{ for all } (c_0, c_1, ..., c_{n-1}) \in C \right\}.$$

The Eulidean hull of C is defined as $Hull(C) = C \cap C^{\perp}$.

1.1. Cyclic codes over $\mathcal{R} = \mathbb{F}_q + v\mathbb{F}_q$.

A linear codes of length n over $\mathcal{R}$ is said to be cyclic if $(c_{n-1}, c_0, ..., c_{n-2}) \in C$ for all $(c_0, c_1, ..., c_{n-1}) \in C$. Each cyclic codes C of length n over $\mathcal{R}$ can be viewed as an ideal of the quotient ring $\mathcal{R}_n = \mathcal{R}[x]/ < x^n - 1 >$, and the corresponding ideal of a cyclic code C has generators of the form

$$< (1 - v)f_1(x), vf_2(x) >,$$

where $f_1(x)$ and $f_2(x)$ are monic divisors of $x^n - 1$ over $\mathbb{F}_q$. Furthermore, $|C| = p^{2n - \deg f_1(x) - \deg f_2(x)}$. In this case, the $q-$dimension of C is $2n - \deg f_1(x) - \deg f_2(x)$.

Let $g(x) = a_0 + a_1 x + ... + a_{k-1} x^{k-1}$ in $\mathbb{F}_q[x]$ a monic polynomial such that a_0 is a unit in $\mathbb{F}_q$. The reciprocal polynomial of $h(x)$ is defined to be $h^* = a_0^{-1} x^{\deg h(x)} h(\frac{1}{x})$, if $h(x) = h^*(x)$, $h(x)$ is called self reciprocal polynomial. Otherwise, $h(x)$ and $h^*(x)$ are called reciprocal polynomial pair. The dual $C^{\perp}$ is generated by

$$< (1 - v)g_1^*(x), vg_2^*(x) >,$$

where $g_1(x) = \dfrac{x^n - 1}{f_1(x)}$ and $g_2(x) = \dfrac{x^n - 1}{f_2(x)}$.

Let n be a positive integer and write $n = p^v \bar{n}$, where $\gcd(\bar{n}, p) = 1$ and v is a nonnegative integer. For coprime positive integers i and j, let $ord_j(i)$ denote the multiplicative order of i modulo j. let

$$N_q = \{k \geq 1 : k|(q^l + 1) \text{ for some } l \in \mathbb{N}\}.$$

By [5, Equation(6)], the polynomial $x^n - 1$ can be factored into a product of monic irreducible polynomials over $\mathbb{F}_q$ of the form

$$(1)\quad x^n - 1 = (x^{\bar{n}} - 1)^{p^v} = \prod_{\substack{j \,|\, \bar{n} \\ j \in \mathcal{N}_q}} \left(\prod_{i=1}^{\gamma(j;q)} g_{ij}(x) \right)^{p^v} \prod_{\substack{j \,|\, \bar{n} \\ j \notin \mathcal{N}_q}} \left(\prod_{i=1}^{\beta(j;q)} f_{ij}(x) f_{ij}^*(x) \right)^{p^v},$$

where

$$\gamma(j;q) = \frac{\phi(j)}{ord_j(q)}, \beta(j;q) = \frac{\phi(j)}{2ord_j(q)},$$

$f_{ij}(x)$ and f_{ij}^* form a reciprocal polynomial pair of degree $ord_j(q)$ and $g_{ij}(x)$ is a self-reciprocal polynomial of degree $ord_j(q)$. Let

$$\mathcal{B}_{\bar{n}} = \deg \prod_{\substack{j \,|\, \bar{n} \\ j \in \mathcal{N}_q}} \left(\prod_{i=1}^{\gamma(j;q)} g_{ij}(x) \right) = \sum_{\substack{j \,|\, \bar{n} \\ j \in \mathcal{N}_q}} \frac{\phi(j)}{ord_j(q)} ord_j(q) = \sum_{\substack{j \,|\, \bar{n} \\ j \in \mathcal{N}_q}} \phi(j).$$

The number $\mathcal{B}_{\bar{n}}$ plays an important role in the study of the average $q-$dimension of the hull of cyclic codes over $\mathcal{R}$.

2. Hull of cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$

We will denote by $\mathcal{C}(n; \mathcal{R})$ the set of all cyclic codes over length n over R. The average q-dimension of the hull of cyclic codes of length n over $\mathcal{R}$ is

$$\mathtt{E}_{\mathcal{R}}(n) = \sum_{C \in \mathcal{C}(n;R)} \frac{\mathtt{dim}_q(Hull(C))}{|\mathcal{C}(n; \mathcal{R})|}.$$

In this section, The characterization of the hulls of cyclic codes of length n over $\mathcal{R}$ is given in terms of their generators. Moreover, we give a formula for the $q-$dimension of the hulls of cyclic codes of length n over $\mathcal{R}$. Finally an explicit formula for $\mathtt{E}_{\mathcal{R}}(n)$ is determined.

Theorem 2.1. *Let $C = (1 - v)C_1 \oplus vC_2$ be a cyclic code of length n over $\mathcal{R}$ generated by $< (1 - v)f_1(x), vf_2(x) >$, where $f_1(x)$ and $f_2(x)$ are monic divisors of $x^n - 1$ over $\mathbb{F}_q$. Then $hull(C)$ is generated by*

$$< (1 - v)\, \mathtt{lcm}(f_1(x), g_1^*(x)), v\, \mathtt{lcm}(f_2(x), g_2^*(x)) >,$$

where $g_1(x) = \dfrac{x^n - 1}{f_1(x)}$ and $g_2(x) = \dfrac{x^n - 1}{f_2(x)}$.

The $q-$dimension of $Hull(C)$ is given in Theorem 2.3 based on the following lemma. The following lemma is required in its proof.

Lemma 2.2. *Let v be a nonnegative integer. Let $0 \leq x, y, z \leq p^v$ be integers. Then the following statements hold.*

(1) $0 \le p^v - \max\{x, p^v - x\} \le \frac{p^v}{2}$.
(2) $0 \le 2p^v - (\max\{y, p^v - z\} + \max\{z, p^v - y\}) \le p^v$.

Theorem 2.3. *Let n be a positive integer and write $n = p^v\bar{n}$, where $\gcd(p, \bar{n}) = 1$ and $v \ge 0$ is an integer. The $q-$dimensions of the hull of cyclic codes of length n over $\mathcal{R}$ are of the form*

$$\sum_{\substack{j \mid \bar{n} \\ j \in \mathcal{N}_q}} ord_j(q).x_j + \sum_{\substack{j \mid \bar{n} \\ j \notin \mathcal{N}_q}} ord_j(q).y_j,$$

where $0 \le x_j \le \gamma(j;q)p^v$ and $0 \le y_j \le 2\beta(j;q)p^v$.

Example 2.4. *Let $n = 33$ and $p = 3$. Then $\bar{n} = 11$ and $v = 1$. The divisors of 11 are 1 and 11.*

(1) *We have $1 \in \mathcal{N}_3$, so $\boldsymbol{ord}_1(3) = 1$ and $\gamma(1;3) = 1$.*
(2) *We have $11 \notin \mathcal{N}_3$, so $\boldsymbol{ord}_{11}(3) = 5$ and $\beta(11;3) = 1$, by Theorem 2.3, the $q-$dimensions of the hulls of cyclic codes of length n over $\mathbb{R}$ are of the form*
$$x_1 + 5.y_{11},$$
where $0 \le x_1 \le 3$ and $0 \le y_{11} \le 6$,
which are $0, 1, 3, 5, 6, 8, 10, 11, 13, 15, 16, 18, 20, 21, 23, 25, 26, 28, 30, 31, 33$.

Lemma 2.5. *Let v be a nonnegative integer and let $0 \le a, b, c \le p^v$ be integers. Then*

(1) $E(\max\{a, p^v - a\}) = \frac{3p^v + 1}{4} - \frac{\delta_{p^v}}{4(p^v + 1)}$ *and*
(2) $E(\max\{b, p^v - c\}) = \frac{p^v(4p^v + 5)}{6(p^v + 1)}$,
where $\delta_{p^v} = 1$ if $v > 0$ and $\delta_{p^v} = 0$ if $v = 0$.

The formula for the average $q-$dimension of the hull of cyclic codes of length n over $\mathcal{R}$ is given as follows.

Theorem 2.6. *Let n be a positive integer and write $n = p^v\bar{n}$, where $\gcd(p, \bar{n}) = 1$ and $v \ge 0$ is an integer. The average $q-$dimensions of the hull of cyclic codes of length n over $\mathcal{R}$ is*

$$\mathtt{E}(n) = n\left(\frac{2}{3} - \frac{1}{3(p^v + 1)}\right) - \mathcal{B}_{\bar{n}}\left(\frac{p^v + 1}{6} + \frac{2 - 3\delta_{p^v}}{6(p^v + 1)}\right).$$

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The Fundamental Mapping Over Group $E_n^{a,b}$

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Abstract. In this work we study the fundamental mappings of group $E_n^{a,b}$[5], group of an elliptic curve defined over ring $A_n = \mathbb{F}_q[\varepsilon];\ \varepsilon^n = 0$, that is given by an homogeneous equation of the form $Y^2Z = X^3 + aXZ^2 + bZ^3$ where $a, b \in A_n$ and $4a^3 + 27b^2$ is invertible in A_n.

Keywords: Elliptic Curves; Cryptography; Generic Group; Finite Field.

THE BI-PERIODIC r-NUMBERS WITH NEGATIVE SUBSCRIPTS

N. ROSA AIT-AMRANE AND HACÈNE BELBACHIR

Abstract. In [2], we have defined a new class of the bi-periodic r-Fibonacci sequence. Then, we introduced a new family of the companion sequences of the bi-periodic r-Fibonacci sequence, named bi-periodic r-Lucas sequence of type s, which generalize the classical Fibonacci and Lucas sequences. Afterwards, we established some properties of these sequences. Here, we propose to extend all the results to the negative subscripts and give more properties of the sequences.

...

2010 Mathematics Subject Classification. 11B39, 11B65, 05A10, 11A15, 11M36.

Keywords and phrases. Bi-periodic Fibonacci sequence, bi-periodic Lucas sequence, generating function, Binet formula, explicit formula, negative indices.

1. Introduction

In [2], we defined a new class of the bi-periodic r-Fibonacci sequence $(U_n^{(r)})_n$ and introduced a new family of companion sequences associated to the bi-periodic r-Fibonacci sequence indexed by the parameter s, with $1 \leq s \leq r$, named the bi-periodic r-Lucas sequence of type s, $(V_n^{(r,s)})_n$. After that, we expressed $V_n^{(r,s)}$ in terms of $U_n^{(r)}$ and s. Actually, first we defined the bi-periodic r-Fibonacci sequence $(U_n^{(r)})_n$

Definition 1.1. *For a, b, c, d nonzero real numbers and $r \in \mathbb{N}$, the bi-periodic r-Fibonacci sequence $(U_n^{(r)})_n$ is defined by, for $n \geq r+1$*

$$U_n^{(r)} = \begin{cases} aU_{n-1}^{(r)} + cU_{n-r-1}^{(r)}, & for \quad n \equiv 0 \pmod 2, \\ bU_{n-1}^{(r)} + dU_{n-r-1}^{(r)}, & for \quad n \equiv 1 \pmod 2, \end{cases} \tag{1}$$

with the initial conditions $U_0^{(r)} = 0, U_1^{(r)} = 1, U_2^{(r)} = a, \ldots, U_r^{(r)} = a^{\lfloor r/2 \rfloor} b^{\lfloor (r-1)/2 \rfloor}$.

Secondly, we introduced a new family of companion sequences related to the bi-periodic r-Fibonacci sequence, called the bi-periodic r-Lucas sequence of type s, $(V_n^{(r,s)})_n$.

Definition 1.2. *For any nonzero real numbers a, b, c, d and integers s, r such that $1 \leq s \leq r$, we define for $n \geq r+1$*

$$V_n^{(r,s)} = \begin{cases} bV_{n-1}^{(r,s)} + dV_{n-r-1}^{(r,s)}, & for \quad n \equiv 0 \pmod 2, \\ aV_{n-1}^{(r,s)} + cV_{n-r-1}^{(r,s)}, & for \quad n \equiv 1 \pmod 2, \end{cases}$$

with the initial conditions $V_0^{(r,s)} = s+1$, $V_1^{(r,s)} = a, V_2^{(r,s)} = ab, \ldots, V_r^{(r,s)} = a^{\lfloor (r+1)/2 \rfloor} b^{\lfloor r/2 \rfloor}$.

The bi-periodic r-Fibonacci sequence $(U_n^{(r)})_n$ and the bi-periodic r-Lucas sequence of type s, $1 \leq s \leq r$ satisfy the following linear recurrence relation.

Theorem 1.3. *For a nonzero real numbers a, b, c, d and s, r such that $1 \leq s \leq r$, the family of the bi-periodic r-Lucas sequence of type s satisfy, for $n \geq 2r+2$*

$$V_n^{(r,s)} = abV_{n-2}^{(r,s)} + (a^{\xi(r+1)}d + b^{\xi(r+1)}c)V_{n-r-1-\xi(r+1)}^{(r,s)} - (-1)^{r+1}cdV_{n-2r-2}^{(r,s)}.$$

After that, we expressed the bi-periodic r-Lucas sequence of type s, $V_n^{(r,s)}$ in terms of $U_n^{(r)}$.

Theorem 1.4. *Let r and s be nonnegative integers such that $1 \leq s \leq r$, the bi-periodic r-Fibonacci sequence and the bi-periodic r-Lucas sequence of type s satisfy the following relationship*

$$V_n^{(r,s)} = \begin{cases} U_{n+1}^{(r)} + sdU_{n-r}^{(r)}, & n \geq r, \quad \textit{for } r \textit{ odd}, \\ U_{n+1}^{(r)} + scbU_{n-r-1}^{(r)} + scdU_{n-2r-1}^{(r)}, & n \geq 2r+1, \quad \textit{for } r \textit{ even}. \end{cases}$$

2. Main results

In this work, we extend these numbers to their terms with negative subscripts. Many results for new forms of these numbers, including Binet formulas, generating functions, and some properties will be presented.

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On a new generalization of geometric polynomials

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Abstract

It is known that the ordered Bell numbers count all the ordered partitions of the set $[n] = \{1, 2, \ldots, n\}$. In this paper, we introduce the deranged Bell numbers that count the total number of deranged partitions of $[n]$. We first study the classical properties of these numbers (generating function, explicit formula, convolutions, etc.). Then we gave an asymptotic formula formula for these numbers

Key words : Ordered Bell numbers, derangements, ordered set partitions, Stirling numbers, Bell numbers.

1 Introduction

A *permutation* σ of a finite set $[n] := \{1, 2, \ldots, n\}$ is a rearrangement (linear ordering) of the elements of $[n]$, and we denote it by

$$\sigma([n]) = \sigma(1)\sigma(2)\cdots\sigma(n).$$

A *derangement* is a permutation σ of $[n]$ that verifies $\sigma(i) \neq i$ for all $(1 \leq i \leq n)$ (fixed-point-free permutation). The derangement number d_n denotes the number of all derangements of the set $[n]$.

A *partition* of a set $[n] := \{1, 2, \ldots, n\}$ is a distribution of their elements to k non-empty disjoint subsets $B_1|B_2|\ldots|B_k$ called *blocks.* We assume that the blocks are arranged in ascending order according to their minimum elements $(\min B_1 < \min B_2 < \cdots < \min B_k)$.

It is well-known that the *Stirling numbers of the second kind*, denoted $\left\{{n \atop k}\right\}$, count the partitions' number of the set $[n]$ into k non-empty blocks. The numbers $\left\{{n \atop k}\right\}$ satisfy the recurrence

$$\left\{{n \atop k}\right\} = \left\{{n-1 \atop k-1}\right\} + k\left\{{n-1 \atop k}\right\} \qquad (1 \leq k \leq n),$$

with $\left\{{n \atop 0}\right\} = \delta_{n,0}$ (Kronecker delta) and $\left\{{n \atop k}\right\} = 0 \quad (k > n)$.

2 The deranged Bell numbers

Now, we introduce the notion of deranged partition and we study the deranged Bell numbers.

Definition 1. A *deranged partition* $\tilde{\psi}$ of the set $[n]$ is a derangement $\tilde{\sigma}$ of the partition $B_1|B_2|\ldots|B_k$, i.e.,

$$\tilde{\psi}([n]) = B_{\tilde{\sigma}(1)}|B_{\tilde{\sigma}(2)}|\ldots|B_{\tilde{\sigma}(k)}$$

such that $B_{\tilde{\sigma}(i)} \neq B_i$ for all $(1 \leq i \leq k)$.

Definition 2. Let $\tilde{F}_n$ be the *deranged Bell number* which counts the total number of the deranged partitions of the set $[n]$.

Proposition 1. *For all $n \geq 0$ we have that*

$$\tilde{F}_n = \sum_{k=0}^{n} d_k \left\{{n \atop k}\right\}. \tag{1}$$

2.1 Exponential generating function

Theorem 1. *The exponential generating function of deranged Bell numbers is given by*

$$\tilde{\mathcal{F}}(t) = \sum_{n\geq 0} \tilde{F}_n \frac{t^n}{n!} = \frac{e^{-(e^t-1)}}{2-e^t}.$$

2.2 Explicit formula

Theorem 2. *For any $n \geq 0$, the sequence $\tilde{F}_n$ can be expressed explicitly as*

$$\tilde{F}_n = \sum_{k=0}^{n} \sum_{i,j=0}^{k} \frac{(-1)^{k+i-j}}{i!} \binom{k}{j} j^n.$$

2.3 Dobiǹski's formula

One of the most important result for Bell number was established by Dobiǹski [2, 4], where he expressed w_n in the infinite series form bellow

$$w_n = \frac{1}{e} \sum_{k\geq 0} \frac{k^n}{k!}.$$

An analogue result for the ordered Bell number was given by Gross [3] as

$$F_n = \frac{1}{2} \sum_{k\geq 0} \frac{k^n}{2^k}.$$

The Dobiǹski's formula for $\tilde{F}_n$ is established by our next theorem

Theorem 3. *For any $n \geq 0$ we have*

$$\tilde{F}_n = \frac{e}{2} \sum_{j\geq 0} \sum_{k\geq j} \frac{(-1)^j}{j!} \frac{k^n}{2^{k-j}}.$$

2.4 Higher order derivatives and convolution formulas

Theorem 4. *For any $m \geq 1$ we have*

$$\tilde{\mathcal{F}}^{(m)}(t) = \tilde{\mathcal{F}}(t) \sum_{i=0}^{m} \sum_{j=i}^{m} (-1)^{i+j} e^{jt} j^{\underline{i}} \left\{ {m \atop j} \right\} \mathcal{F}^i(t), \tag{2}$$

where $\tilde{\mathcal{F}}^{(m)}(t)$ is the m^{th} derivative of $\tilde{\mathcal{F}}(t)$.

We will use the classical singularity analysis technic (see Chapter 5 of [6]) to deduce the asymptotic behavior a sequence a_n using the singularities of its generating function $\mathcal{A}(t)$.

Theorem 5. *The asymptotic behavior $\tilde{F}_n$ is*

$$\frac{\tilde{F}_n}{n!} \sim \frac{1}{2e \log^{n+1}(2)} + O\left(6.3213\right)^{-n}, \qquad n \longrightarrow \infty.$$

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Boundedness of the Numerical Range

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Abstract

This paper is concerned with the numerical range and some related properties of the operator self-adjoint A, where A is bounded linear operator.

We show the following results:

(i) If A is a self-adjoin operator, then: $\forall n \in \mathbb{N} : W(A^n) \subseteq [W(A)]^n$.

(ii) Determine the smallest convex subset containing the numerical range of a bounded linear operator on a *Hilbert* space.

(iii) Determine necessary and sufficient conditions such that the numerical range of a bounded linear operator $A \in B(H)$ is a line segment, and we present a new set S, such that S is a set of operators whose the numerical range $W(A)$ is a line segment.

keywords: Numerical Range, Numerical Radius.

Probability and Statistics

A MAXIMUM PRINCIPLE FOR MEAN-FIELD STOCHASTIC DIFFERENTIAL EQUATION WITH INFINITE HORIZON

ABDALLAH ROUBI AND MOHAMED AMINE MEZERDI

Abstract. We consider an infinite horizon optimal control of a system where the dynamics evoves according to a mean-field stochastic differential equation and the cost functional is also of mean-field type. These are systems where the coefficients depend not only on the state but also on its marginal distribution via some linear functional. Under some concavity assumptions on the coefficients as well as on the Hamiltonian, we are able to prove a verification theorem, which gives sufficient condition for optimality for a given admissible control. In the absence of concavity, we prove a necessary condition for optimality in the form of a weak Pontriagin maximum principle, given in terms of stationarity of the Hamiltonian.

2010 Mathematics Subject Classification. 60H10, 60H07, 49N90

Keywords and phrases. Mean-field stochastic differential equation, Infinite horizon, Stochastic maximum principle, Backward stochastic differential equation, Adjoint process, Hamiltonian.

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A WALSH-FOURIER ANALYSIS OF SCHIZOPHRENIC PATIENTS' BRAIN FUNCTIONAL CONNECTIVITY

HOUSSEM BRAIRI AND TAREK MEDKOUR

ABSTRACT. We consider the problem of analyzing the brain functional connectivity of schizophrenic patients, using Walsh-Fourier analysis.To this end, we propose three tests: the first one is based on the cumulative Walsh periodogram, and is shown to converge to a Brownian bridge. The second test is based on applying the Cramer–von Mises functional to an estimate of the Walsh spectral density, and is shown to converge to a Normal distribution, while the last test is based on a distance to whiteness, and is shown to have an approximate scaled chi-square distribution. Simulations are reported on the performance of the tests.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 62M10, 60H40, 62M15.

KEYWORDS AND PHRASES. Discrete-valued time series, Walsh-Fourier Analysis, White noise.

1. DEFINE THE PROBLEM

In view of their important role, discrete-valued time series is attracting considerable attention due to its many uses in several studies such as EEG sleep patterns, gene characterization, geomagnetic reversals in the polarity of the earth $\cdots$ etc. In this work, we consider a problem which is of fundamental importance and has not been addressed yet, namely testing a discrete-valued time series for whiteness in the sequency domain. The problem is formulated as follows: Let $X(0), X(1), \cdots, X(N-1)$ be a sample from a zero mean, second-order stationary discrete-valued time series $\{X(n), n = 0, \pm 1, \ldots\}$, of length $N = 2^p$ $(p \in \mathbb{N}^*)$. Let $M = 2^s$ $(s \in \mathbb{N}^*)$ with $M \ll N$ and $\Gamma(j) = E\{X(n)X(n+j)\}$ be the autocovariance function. We consider the null hypothesis H_0 that $X(n), n = 0, \cdots, N1$, is a white noise. Equivalently, in terms of the Walsh spectral density function $F\lambda)$, the null hypothesis becomes:

$$H_0 : F(\lambda) = \Gamma(0) \ \forall \lambda \qquad \text{vs} \qquad H_1 : \exists \lambda', F(\lambda') \neq \Gamma(0)$$

The analysis of Schizophrenic Patients' Brain Functional Connectivity is based on the test statistics which are proposed in this work.

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A NEW THRESHOLD MODEL FOR FINANCIAL TIME SERIES ANALYSIS

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Abstract. In this communication, we introduce a new threshold model to analyze the stochastic volatility of financial time series. Being inspired from the threshold stochastic volatility model proposed by Breidt (1996), our model may explain better more nonlinear stylized facts observed in financial assets namely asymmetry and leverage effect. Keeping the piecewise-linear structure in the log-volatility as in Breidt (1996), we introduce a new flexible regime-switching mechanism based on the buffered process introduced in Li et al. (2015). This transition mechanism avoids the sudden jump in the log-volatility imposed by the classical model and allows a smooth transition between regimes. We provide a Sequential Monte Carlo method to estimate the model's parameters. We applied our model to fit the Honeywell International Inc (HON) index.

2010 Mathematics Subject Classification. 62M10, 37M10, 91B84.

Keywords and phrases. stochastic volatility, threshold time series, buffer zone .

1. Define the problem

Threshold models introduced in [5], has been adopted to reproduce asymmetric behavior and to capture the leverage effect exhibited by financial returns. The use of threshold models to explain asymmetric volatility has been, firstly, known in the deterministic ($ARCH/GARCH$-type) volatility modeling framework (see e.g, [7], [8]). This approach has been extended in [2] to the stochastic volatility analysis area, by the definition of the Threshold Stochastic Volatility model (TSV). Based on Tong's pioneering work, Breidt's model assumes that, according to whether the information good or bad, the volatility dynamics follows a two-regime threshold model, where the log-volatility in each regime is represented by a first order autoregressive model and the transition between regimes is a function of lagged stock returns signs. Although threshold models have been hugely successful, It was pointed out that they have poor performance around the boundaries between the different regimes. This is mainly due to the sudden change in the probabilistic structure of the model, which may not be the case in the real world [6]. In fact, empirical studies have shown that asset prices have a particular volatility reaction for good and bad news [1]. Assuming the threshold effect in the volatility and without loss of generality, for a two-regime situation, there are some cases where the shift of regime may not happen at the same threshold value. In fact, when the return on the price of an asset exceeds a particular positive threshold r_U, the market can affirm the advent of good news, while the bad news is only confirmed when return crosses another negative threshold, r_L. The interval $(r_L; r_U]$ serves therefore as a buffer zone, no information comes in when the return falls within this zone and volatility structure is also assumed to remain unchanged. It is clear that this finding cannot be captured by the classical threshold model since it considers only a single threshold parameter. This situation has been discussed and investigated in the introductory paper untitled "Hysteresis autoregressive time series model" of [3] where a new more flexible regime switching mechanism is defined. This mechanism is based on the replacement of the single threshold parameter by the zone $(r_L; r_U]$ which is called "Hysteresis" or "Buffer" zone. This approach provides a rigorous definition of the regime indicator. This idea has already been adopted for the deterministic volatility modeling (see e.g, [4], [9]).

In this communication, we introduce this approach in the SV modeling context. By considering the same new flexible regime-switching mechanism, we define a new threshold stochastic volatility model that we call Buffered Stochastic Volatility (BSV) model. We employ a Sequential Monte Carlo method to provide a Maximum Likelihood estimate of the proposed model. The performance of the proposed estimation method is discussed through a simulation study and an application for fitting the Honeywell International Inc (HON) index.

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AEROSOLS AGGREGATION MODELING BASED ON NUMERICAL SIMULATION OF SMOLUCHOWSKI EQUATIONS

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Abstract. Atmospheric aerosols represent a complex dynamic mixture of microscopic solid or liquid droplets particles suspended in atmosphere. Atmospheric particles come from many different sources consisting of both natural origins and/or anthropogenic activities. Aerosols influence the energy balance of the Earth. In fact, interaction procedure between solar/terrestrial radiation fluxes and atmospheric aerosols play a primary role in affecting the Earth's climate by scattering light and changing Earth's reflectivity. Moreover, aerosols are of central importance for cloud formation. Hence, aerosols alter planetary albedo by affecting cloudiness and global average temperature. The aim of this work is to study the atmospheric aerosols coagulation process greatly enhanced by the Van der Waals forces and monitored by the Brownian motion. We analyse an approach for solving Smoluchowski's coagulation equation employing the Monte Carlo probabilistic method based on the use of random numbers in repeated experiments. Additionally, several numerical simulations have been implemented and evaluated regarding their Central Processing Unit (CPU) times and their accuracy in terms of mass concentrations. All our numerical tests show that the numerical solutions calculated by MC algorithms converges to the exact solutions.

2010 Mathematics Subject Classification. 65C05, 65C10, 65C30, 65C35, 65C50, 34K50.

Keywords and phrases. Computational probability, Aerosols statistical approach, Stochastic differential equations, Monte Carlo method.

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In this work, we deal with a stochastic control problem for a backward doubly stochastic differential equation governed by a standard Wiener motion and a fractional Brownian motion with Hurst parameter greater than half. We explicit the adjoint process and derive the stochastic maximum principle using two major approaches: the first one is the Doss-Sussmann transformation and the second one is the Malliavin calculus. The set of the control domain is convex. The criterion to be minimized is in the general form, with initial cost.

ASYMPTOTIC ANALYSIS OF A KERNEL ESTIMATOR OF TREND FUNCTION FOR STOCHASTIC DIFFERENTIAL EQUATION WITH ADDITIVE A SMALL WEIGHTED FRACTIONAL BROWNIAN MOTION

ABDELMALIK KEDDI, FETHI MADANI, AND AMINA ANGELIKA BOUCHENTOUF

Abstract. In this work, we consider the problem of nonparametric estimation of trend function for stochastic differential equations driven by a small weighted fractional Brownian motion (weighted-fBm). Under some general conditions, the consistent uniform, the rate of convergence as well as the asymptotic normality of our estimator are established.

2010 Mathematics Subject Classification. 62M09, 60G15

Keywords and phrases. Weighted fractional Brownian motion; trend function; kernel estimator; stochastic differential equations; nonparametric estimation.

1. Introduction

In this work, we investigate the problem of estimating the trend function $S_t = S(x_t)$ for process satisfying stochastic differential equations of the type

$$dX_t = S(X_t)dt + \varepsilon dB_t^{a,b},\ X_0 = x_0,\ 0 \leq t \leq T, \tag{1}$$

where $\left\{B_t^{a,b}, t \geq 0\right\}$ is a weighted fractional Brownian motion with known parameters a and b, such that $a > -1$, $0 < b < 1$, $b < a+1$ and $a + b > 0$. We estimate the unknown function $S(x_t)$ by a kernel estimator $\hat{S}_t$ and obtain the asymptotic properties as $\varepsilon \longrightarrow 0$.

Using the method developed in [3]. Then, the kernel estimator of S_t is given by

$$\hat{S}_t = \frac{1}{\phi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dX_\tau, \tag{2}$$

where $G(u)$ is a bounded function with finite support $[A, B]$

2. Main result

We suppose that the function $S : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following assumptions:

(**A1**) There exists $L > 0$ such that

$$\left|S(x) - S(y)\right| \leq L\left|x - y\right|,\ 0 \leq t \leq T. \tag{3}$$

(**A2**) There exists $M > 0$ such that

$$|S(x)| \leq M(1+|x|),\ x \in \mathbb{R},\ 0 \leq t \leq T.$$

(**A3**) Assume that the function $S(x)$ is bounded by a constant C.

We suppose also that $G(u)$ is a bounded function with finite support $[A,B]$ satisfying the following hypotheses

(H1) $G(u) = 0$ for $u < A$ and $u > B$, and $\int_A^B G(u)du = 1$,

(H2) $\int_{-\infty}^{+\infty} G^2(u)du < \infty$,

(H3) $\int_{-\infty}^{+\infty} u^{2(k+1)} G^2(u)du < \infty$,

(H4) $\int_{-\infty}^{+\infty} |G(u)|^{\frac{2}{a+b+1}} du < \infty$,

(H5) $\phi_\varepsilon \longrightarrow 0$ and $\varepsilon^2 \phi_\varepsilon^{-1} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

(H6) $\int_{-\infty}^{+\infty} u^j G(u)du = 0$ for $j = 1, 2, ..., k$,

(H7) $\int_{-\infty}^{+\infty} u^{k+1} G(u)du < \infty$; and $\int_{-\infty}^{+\infty} u^{2(k+2)} G^2(u)du < \infty$.

⋇ **Uniform convergence**

Theorem 2.1. *Suppose that the assumptions **(A1)-(A3)** and **(H1)-(H5)** hold true. Then, for any* $0 < c \leq d < T$, $a > -1$, $0 < b < 1$, $b < a+1$, *and* $a + b > 0$, *the estimator* $\hat{S}_t$ *is uniformly consistent, that is,*

$$\lim_{\varepsilon \longrightarrow 0} \sup_{S(x)\in\Sigma_0(L)} \sup_{c\leq t\leq d} \mathbb{E}_S(|\hat{S}_t - S(x_t)|^2) = 0.$$

⋇**The rate of convergence**

Theorem 2.2. *Suppose that* $a > -1$, $0 < b < 1$, $b < a+1$, $a+b > 0$, *and* $\phi_\varepsilon = \varepsilon^{\frac{2}{2k-a-b+3}}$. *Then, under the hypotheses **(A1)-(A3)** and **(H1)-(H7)**, we have*

$$\limsup_{\varepsilon \longrightarrow 0} \sup_{S(x)\in\Sigma_k(L)} \sup_{c\leq t\leq d} \mathbb{E}_S(|\hat{S}_t - S(x_t)|^2)\varepsilon^{\frac{-4(k+1)}{2k-a-b+3}} < \infty.$$

⋇ **The asymptotic normality**

Theorem 2.3. *Suppose that* $a > -1$, $0 < b < 1$, $b < a+1$, $a+b > 0$, *and* $\phi_\varepsilon = \varepsilon^{\frac{2}{2k-a-b+3}}$. *Then, under the hypotheses **(A1)-(A3)** and **(H1)-(H7)**, we have*

$$\varepsilon^{\frac{-2(k+1)}{2k-a-b+3}} \left(\hat{S}_t - S(x_t)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(m, \sigma_{a,b}^2), \ \text{as} \ \varepsilon \longrightarrow 0,$$

where

$$m = \frac{S^{k+1}(x_t)}{(k+1)!} \int_{-\infty}^{+\infty} G(u)u^{k+1}du,$$

and

$$\sigma_{a,b}^2 = b \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} G(u)G(v)(u \wedge v)^a (u \vee v - u \wedge v)^{b-1} du dv,$$

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ASYMPTOTIC PROPRIETES OF NON PARAMETRIC RELATIVE REGRESSION ESTIMATOR FOR ASSOCIATED AND RANDOMLY LEFT TRUNCATED DATA

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Abstract. Let Y be a real random variable of interest and X an R^d-valued random vector of covariates. We want to estimate Y after observing X. There are several ways to estimate it and one of the most popular is the regression method which modeled by the following consideration: $Y = m(X) + \epsilon$, where m is the unknown regression function and ϵ is a random error variable.
Classicaly, the regression function m is estimated by using the mean squared error as a loss function. However, this loss function is very sensitive to outliers. One of the techniques we can use to overcome this problem is to use alternative loss function based on the squared relative error. Relative error estimation has been recently used in regression analysis (see Park and Stefanski (1998), Jones et al.(2008)). This technique is useful in analyzing data with positive responses, such as life times which we interest in this work. Another particularity of the life times is that are often not observed completely. Censored and truncation are the most current forms of the incomplete data. In this work we are interested in the left truncated data, where the observation (X, Y) is interfered by another independent rv T such that all three random quantities Y, X, and T are observable only if $Y \geq T$. This model is originally appeared in astronomy (woodroofe (1985)), then extend to several domains as economics, epidemiology, demographics, actuarial. When we use the least square error as a loss function to determine the regression function m, Ould Saïd and Lemdani (2006) built a kernel type estimator of $m(x)$ which take into account the truncation effect. Following the same arguments, we define the kernel estimator of the truncated relative error regression of m and we study its asymptotic proprieties. We give also illustrations of the results on simulated data.

2010 Mathematics Subject Classification. xxxx, xxxx, xxxx.

Keywords and phrases. Association, Random left-truncation (RLT) model, Kernel estimator, Nonparametric regression, Rate of convergence, Relative error, Strong consistency.

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BERNSTEIN-FRECHET INEQUALITIES FOR NOD RANDOM VARIABLES AND APPLICATION TO AUTOREGRESSIVE PROCESS

CHEBBAB IKHLASSE

ABSTRACT. In this paper, we establish exponential inequalities for NOD random variables that allow use to create a confidence interval for the parameter of the first order autoregressive process. Using these inequalities, we show the almost complete convergence for the estimator of this parameter.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 60B, 60F05, 60F15, 60F17, 60G10.

KEYWORDS AND PHRASES. Autoregressive process, Convergence, Exponential inequalities, NOD random variables.

1. INTRODUCTION

The autoregressive process takes an important part in predicting problems leading to decision making. Let us consider the autoregressive process order 1 defined by

$$Y_k = \theta Y_{k-1} + \xi_k$$

Where θ is the autoregressive parameter and where $(\xi_k)_k$ is a sequence of normally distributed random variables, with zero mean. Consider Y_{k-1} as NOD output variables. We use the least sequares methode to estimate the parameter θ.

Lehmann [3] introduced a simple and natural definition of negative dependence: A sequence $\{Y_i, 1 \leq i \leq n\}$ of random variables is said to be pairwise negative quadrant dependent (pairwise NQD) if for any real y_i, y_j and $i \neq j$, $\mathbf{P}(Y_i > y_i, Y_j > y_j) \leq \mathbf{P}(Y_i > y_i)\mathbf{P}(Y_j > y_j)$. The concept of negatively orthant dependent random variables was introduced by Ebrahimi and Ghosh [2] as follows.

Definition 1.1. *The sequence $\{Y_n, n \geq 1\}$ of random variables are said to be lower negatively orthant dependent (LNOD), if for any $n \geq 1$*

$$\mathbf{P}(Y_1 \leq y_1, Y_2 \leq y_2, ..., Y_n \leq y_n) \leq \prod_{j=1}^{n} P(Y_j \leq y_j).$$

for every $y_1, ..., y_n \in \mathbb{R}$
The sequence $\{Y_n, n \geq 1\}$ of random variables are said to be upper negatively orthant dependent (UNOD), if for any $n \geq 1$

$$\mathbf{P}(Y_1 > y_1, Y_2 > y_2, ..., Y_n > y_n) \leq \prod_{j=1}^{n} P(Y_j > y_j).$$

for every $y_1, ..., y_n \in \mathbb{R}$

The sequence $\{Y_n, n \geq 1\}$ *of random variables are said to be negatively orthant dependent (NOD) if* $\{Y_n, n \geq 1\}$ *are both LNOD and UNOD.*

2. Main results

Theorem 2.1. *For any* $\epsilon < \frac{1}{4}\log 3$ *positive and for R rather large, we have*

$$\mathbf{P}(\sqrt{n}|\theta_n - \theta| > R) \leq 2\exp(-\tfrac{1}{2}R^2\epsilon^2 A_1) + 3^{-(n-1)/4}\exp(n\epsilon) \quad (1)$$

Where $A_1 = \dfrac{1}{\gamma^2 L^2}$ *is a positive constant.*

Corollary 2.2. *The sequence of estimators* $(\theta_n)_{n\in\mathbb{N}}$ *converges almost completely to the parameter* θ *of the autoregressive process of order 1.*

Corollary 2.3. *The inequalities (1) give us the possibility to construct a confidence interval for the parameter* θ*.*

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Asymptotic Normality of the Kernel Regression Estimator with Associated and LTRC Data

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We propose a kernel nonparametric estimator of the regression function for incomplete data. The incompleteness model studied here is the left truncated and right censored (LTRC) one. We suppose that the observations are independent and identically distributed (iid). A simulation study is carried out in order to show the performance of our estimator.

Keywords: Non parametric regression, Kernel estimator, Truncated-censored data

DECONVOLVING THE DISTRIBUTION FUNCTION FROM ASSOCIATED DATA: THE ASYMPTOTIC NORMALITY.

BEN JRADA MOHAMMED ESSALIH AND DJABALLAH KHEDIDJA

Abstract. In reliability theory or survival analyses, it is common to observe data that are not only contaminated but weakly dependent too. The goal here is to discuss the problem of estimating the unknown cumulative density function $F(x)$ of X when only corrupted observations $Y = X + \varepsilon$ are present, where X and ε are independent unobservable random variables and ε is a measurement error with a known distribution. For a sequence of strictly stationary and positively associated random variables and assuming that the tail of the characteristic function of ε behaves either as super smooth or ordinary smooth errors, we obtain the asymptotic normality.

2010 Mathematics Subject Classification. 62G05, 60G10, 62G20.

Keywords and phrases. cumulative density function, Positively Associated, Deconvolution, Asymptotic Normality.

1. Define the problem

We consider the problem of estimation from observations that are contaminated by additive noise $\{\varepsilon_i\}_{i=1}^n$. Due to the nature of the experimental environment or the measuring tools, the random process $\{X_i\}_{i=1}^n$ is not available for direct observation. Instead of X_i, we observe the random variables Y_i given by

$$Y_i \triangleq X_i + \varepsilon_i, \quad i = 1, ..., n. \tag{1}$$

The focus is to estimate nonparametrically the unknown common cumulative density function (c.d.f.) $F(x)$ of a process $\{X_i\}_{i=1}^n$ which is assumed to be strictly stationary and positively associated. In addition, we assume that the density function (p.d.f.) $f(.)$ of the process $\{X_i\}_{i=1}^n$ exists. Furthermore, the noise process $\{\varepsilon_i\}_{i=1}^n$ consists of independent and identically distributed (i.i.d.) random variables, and independent from $\{X_i\}_{i=1}^n$, with known density function $r(.)$. Thus the common probability density function $g(.)$ of the random variables Y_i is given by:

$$g(x) = \int_{-\infty}^{+\infty} f(x - t) r(t) dt. \tag{2}$$

Model (1) is called a convolution and the problem of estimating f with this model occurs in various domains. This model has been studied in Experimental Sciences. For example, Biological Organisms, Communication Theory, and Applied Physics.

The literature abounds of work devoted to the study of the p.d.f. in convolution problems. [4] proposed a consistent estimator for the density based on grouped data for some cases of error density. [5] considered the estimation of the multivariate probability density functions under some structures of dependence. [7] used the Moving Polynomial Regression (MPR) to smooth the empirical distribution function estimator. [6] considered the asymptotic uniform confidence bands.

The c.d.f. deconvolution has not attracted as many research. [8] developed the approach to examining the estimation of the c.d.f. and treated its corresponding asymptotic normality in the case where the joint random process $\{X_i, \varepsilon_i\}_{i=1}^{n}$ is stationary and satisfies the ρ-mixing condition and fulfilling some additional assumptions. Furthermore, the contaminated noises $\{\varepsilon_i\}_{i=1}^{n}$ are assumed to have a dependence structure and are either ordinary smooth or super smooth. [9] studied the minimax complexity of this problem when the unknown distribution has a density belonging to the Sobolev class and the error density is ordinary smooth. [10] considered the deconvolution when the unknown distribution is modeled as a mixture of p known distributions. [11] studied a consistent estimator of a distribution function from observations contaminated with additive Gaussian errors. Fan (1991) considered the estimate based on integration of the density deconvolution estimator. [12] developed the estimation of the c.d.f. in the case where data are corrupted by heteroscedastic errors.

We study the quadratic mean convergence and deduce the mean-square convergence rate for the deconvolving cumulative density estimator under various assumptions on the characteristic function ϕ_r of the measurement error. The following two cases are generally distinguished:

- ϕ_r decays algebraically at infinity

$$|t|^{\beta} |\phi_r(t)| \underset{|t|\to+\infty}{\to} \beta_1 \text{ for some } \beta > 0 \text{ and } \beta_1 > 0.$$

In this case, the error is called ordinary smooth.

- ϕ_r decays exponentially fast at infinity

$$\beta_2 e^{-m|t|^{\alpha}} |t|^{\beta} \leq |\phi_r(t)| \leq \beta_3 e^{-m|t|^{\alpha}} |t|^{\beta},$$

  for some positive constants α , m, real β, and positive constants β_2 and β_3.

This is called supersmooth error.

The parameter β is called the order of the noise density $r(x)$. Actually, it has a direct impact on the rate of convergence of the estimate $F_n(x)$. Particular examples of supersmooth distribution are Normal, Mixture Normal, Cauchy densities $r(x)$. The ordinary smooth distribution covers in particular the case of Gamma, Double Exponential, and Symmetric Gamma densities $r(x)$.

Next, it is of practical interest to show that the deconvolution difficulties are heavily related to the smoothness of the error distribution. Indeed, super smooth distributions are more difficult to deconvolve than ordinary smooth distributions, see for example the proofs in [13].

The infinite random process $\{X_i\}_{i=1}^{+\infty}$ is positively associated (PA for short), or just associated, if every finite subcollection $\{X_i\}_{i=1}^{n}, n \geq 1$ satisfies the property given in the following definition.

Definition 1. *A finite family of random variables* $\{X_i\}_{i=1}^n$ *is said to be positively associated if*

$$Cov\left[\Phi_1(X_i, i \in A_1), \Phi_2(X_j, j \in A_2)\right] \geq 0,$$

for every pair of disjoint subsets A_1 *and* A_2 *of* $\{1, 2, ..., n\}$*, and* Φ_l *are coordinatewise increasing functions and this covariance exists for* $l = 1, 2$.

Definition 1, which was introduced by [1], includes several mixing process classes. Note that associated processes have attracted a lot of research attention since they arise in a variety of contexts. For instance, in Finance (see [2]), and in Applied physics (see [3]), and even in Percolation theory. We may also cite the homogeneous Markov chains as a direct example of the association property and normal random vectors with nonnegative covariance sequences.

It is worthy to note that, if the underlying process $\{X_i\}_{i=1}^n$ is associated, then the process $\{Y_i\}_{i=1}^n$ involving the convolution model in (1) is a corrupted-associated random process. Actually, from Property P2 of [1] (mentioned later), the independence between the processes $\{X_i\}_{i=1}^n$ and $\{\varepsilon_i\}_{i=1}^n$ ensures the association of the union $\{X_i\}_{i=1}^n \cup \{\varepsilon_i\}_{i=1}^n$. Fortunately, all dealing here is with a strictly stationary process. In fact, and as mentioned above, $\{\varepsilon_i\}_{i=1}^n$ consists of i.i.d. rvs. Since $\{X_i\}_{i=1}^n$ are independent from $\{\varepsilon_i\}_{i=1}^n$, it is clear that $\{Y_i = X_i + \varepsilon_i\}_{i=1}^n$ is a strictly stationary random process.

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DIFFUSION APPROXIMATION OF A FINITE-SOURCE M/M/1 RETRIAL QUEUEING SYSTEM

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Abstract. A single server retrial queue with finite number of sources is simply described as a mathematical model. We establish a diffusion approximation of the scaled number of requests in the orbit using the convergence of the generator approach.

2010 Mathematics Subject Classification. 60K25, 68M20, 90B22.

Keywords and phrases. Convergence of the Generator Approach, Diffusion Process, Finite-Source Retrial Queue, Infinitesimal Generator, Scaled Number of Blocked Requests.

1. Define the problem

Diffusion approximation consists of scaling down the discret space-state of a Markov process by a quantity that tends to infinity and then identifying the resulting diffusion process, which is a Markov process defined on a continuous space-state. In the paper [1], the author has demonstrated that the virtuel waiting time process in in the M(t)/G/1/∞ queue with FIFO discipline converges to a Brownian motion in heavy traffic. Whereas, in [2],it is shown that the number of sources or retrials in an M/M/c retrial queue is approximated by an Ornstein-Uhlenbeck process in a steady state. In this work, we are going to show that the number of blocked requests in an M/M/1/N retrial queue can be approximated by a certain diffusion process under an asymptotic regime. As in [1] and [2], the diffusion approximation is elaborated using the convergence of the generator approach (for details see [3]) after the definition of the diffusion scaled process (see [4]).

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Development of Artificial Neural Network Time Series Model with Stochastic Optimization for the Prediction of Daily Solar Radiation in Oran

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Abstract:

In this study, we present an application of Artificial Neural Networks (ANNs) in the renewable energy domain. In particular, we focus on the Multi-Layer Perceptron (MLP) network, which has been the most widely used ANNs architectures in both the renewable energy and time series forecasting domains. We have developed an ANN model time series for the daily prediction of global solar radiation whit an automatic selection of the optimal architecture of the ANN model; depending on the training data. Thus, the stochastic optimization algorithm Adam (The Adaptive Moment Estimation Optimizer) has been adopted to adjust the training parameters. A thirty-nine years reanalysis data series between 1980 and 2018 was used for training and implementation of the model, and validation was carried out with respect to the year 2019. The results of the error analysis obtained show that the model developed has a good performance in the line with the previous studies. Thus, we notice that the consideration of seasonality slightly improves the accuracy of the forecasts. The best-chosen ANN model is identified on the basis of the minimum mean absolute error (MAE) and root mean square error (RMSE). In addition, this study confirms that the accuracy of ANN model predictions depends on the complete set of data used to build the network of the intended application. The ANN model developed is characterized by reasonable daily prediction accuracy, with a low RMSE of $3.112\ Mj/(m^2.d)$. *which verifies the accuracy and ability of the model to predict solar radiation in order to ensure an optimal management of solar energy farms, where meteorological data measurement facilities are not in place in Oran.*

Keywords: ANN, Multi-Layer Perceptron; ANN Time Series model; Solar Radiation; Daily Forecast.

Abstract

The paper deal with the robust nonparametric regression for a functional single index covariates when the response variables are missing at random (MAR), for both cases, without and with unknown scale parameter. We establish, the almost complete convergence rate of the proposed estimators. Some simulations study are drawing, and real data analysis are given to illustrate the higher predictive performances of our proposed method.

Keywords {robust regression , functional single index covariate , almost complete convergence , missing data , scale parameter.}

ESTIMATION METHODS FOR PERIODIC $INAR(1)$ MODEL WITH GENERALIZED POISSON DISTRIBUTION

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Abstract. This communication deals with the parameters estimation problem of a *Periodic Integer-valued Generalized Poisson* $AR(1)$ model which has been shown to be useful to describe overdispersion and underdispersion encountered in periodically correlated integer-valued time series. Some probabilistic and statistical properties are established. Indeed, the periodically correlated stationarity conditions, in the first and the second moments are provided. Moreover, the structure of the periodic autocovariance is obtained. The estimation problem is addressed through *the Yule-Walker* (YW), *the Conditional Least Squares* (CLS) and *the Conditional Likelihood* (CML) methods. The performance of these methods is done through an intensive simulation study.

2010 Mathematics Subject Classification. 62F12, 62M10

Keywords and phrases. Generalized Poisson distribution, periodically correlated integer-valued process, periodic integer-valued, autoregression $PINAR$ model. Periodically stationarity conditions.

1. Define the problem

Integer-valued time series arise in many practical settings. Count series are non-negative integers and are usually correlated over time. Many of data sets are characterized by low counts, over dispersion, under dispersion, ruling out normal approximation and they can not be well approximated by continuous variables. Modeling and analyzing counts series remains one of the most challenging and undeveloped areas of time series analysis, so it is necessary to develop an appropriate modelling strategy.

One of the approaches developed is based on a random operation called *thinning operation* capable of preserving the integer valued nature of the variables, giving rise to the class of Integer valued Autoregressive, $INAR$, models. In first, researchers use the Poisson distribution as an integral feature of the process. McKenzie (1985) and, independently, Al-Osh and Alzaid (1987) introduced the first-order integer-valued autoregressive, $INAR(1)$, model based on *the binomial thinning operator,* Steutel and Van Harn(1979), and dealt with discrete time stationary processes with Poisson marginal distributions, called Poisson $INAR(1)$ process. Issues such as inference and forecasting for Poisson time series models have been discussed.

In practice, however, the Poisson distribution is not always suitable for modeling because it characterized by the property of *equidispersion (i.e.,the mean is equal to the variance),* so if we have one observation underdispersed or overdispersed *(i.e.,the mean is smaller or greater than the variance)*, the Poisson distribution should not be applied. In such cases, there are several alternative models have been proposed in the literature. Thus, Bourguinon and *al.* (2018) proposed two new binomial thinning $INAR(1)$ process with Double Poisson (DP) and Generalized Poisson (GP) innovations, denoted by $INARDP(1)$ and $INARGP(1)$, respectively, for modeling non-negative integer time series with equidispersion, underdispersion and overdispersion.

It is recognized nowadays, that many integer-valued time series encountered in various fields as the environmental, economic particularly financial ones, exhibit a periodic feature, in their autocovariance structure (as examples *Number of cases of campylobacterosis infections time series* studied by Ferland *et al* (2006), *Monthly number of short-term unemployed people in Penamacor County Portugal,* studied by Monteiro *et al* (2010), *Monthly number counts claims of short-term disability benefits,* studied by Bourguignon *et al* (2015) and independently by Zhu and Joe (2006) and Freeland (1998)), that cannot be encountered by the standard integer-valued modeling. The periodic, in time, coefficient models are very adequat for modeling these *Periodically Correlated Processes*, in the sense of Gladyshev (1963).

In this communication, we propose a first order Periodic Integer-Valued Autoregressive $PINAR(1)$ with Generalized Poisson marginal distribution, our new model can fit the data with periodically correlated structure, and handle the underdispersed, equidispersed and overdispersed situations. The conditions of stationarity of the first and second order moments are established. The periodic autocovariance structure of the proposed model is, under these conditions, established. Moreover, the explicit forms of the mean and the variance are driven. The performance of the three different presented methods namely *the Yule-Walker* (YW)*, the Conditional Least Squares* (CLS) and *the Conditional Likelihood* (CML) methods are studied via intensive simulation studies.

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Existence of optimal relaxed controls for mean-field stochastic systems

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Abstract

We study the existence of an optimal control for systems governed by stochastic differential equations of mean-field type. In these equations, the drift and the diffusion coefficient depend not only on the state of the system, but also on the expectation of some function of the state. For nonlinear systems, we prove the existence of an optimal relaxed control, by using tightness techniques and Skorokhod selection theorem. In the case where the coefficients are linear maps and the cost functions are convex, we prove by using weak convergence techniques the existence of an optimal strict control, adapted to the initial filtration.

Keywords: Mean-field stochastic differential equation; relaxed control; strict control; weak convergence; tightness.

2010 Mathematics Subject Classification. 60H10, 60H07, 49N90

FDA : LOCAL LINEAR MODE REGRESSION

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Abstract. This paper is devoted to study the asymptotic properties of local linear mode regression function in iid setting and for functional data.

2010 Mathematics Subject Classification. 62G05, 62G08, 62G20.

Keywords and phrases. functional data, locally modelled regression, mode regression.

1. Define the problem

In nonparametric statistics, many studies are carried out to give powerful tools to model and study the relationship between the response variable. The nonparametric mode regression function has long been a question of great interest in a wide range of fields for instance in econometrics, biologie, astronomy...
In most phenomenon, the observations of data have functional nature. Indeed, the technology's advance contributes by providing many studies in different fields with modern and relevant measuring instruments. In other side many statistical problems appear such "the strong correlation between the variables, the ratio between the number of variables and the size of sample"(see [10]). Therefore, the functional data analysis (FDA) appears to model and treat such kind of data, for an updated of references, we can lead the reader to the monographs by [17], [18], [3] and [10].
For more than decades, many papers relied on the kernel method to estimate the nonparametric regression function (see : [19], [16] and [11]). Then, [9] generalized the kernel regression estimator of Nadaraya-Watson familiar function where this model was adopted in many studies to find more asymptotic results such as : the k-nearest-neighbours (k-NN) estimator is investigated by [4], convergence in $\mathbb{L}^2$ norm (see : [5]) which is generalized in the α-mixing case (see [6]) and recently, [13] stated the uniform in bandwidth for kernel regression estimator.
However, the previous literatures used the Nadaraya-Watson techniques as estimation method which has some drawbacks, mainly, in the bias term. Hence, in the functional data setup, the local linear method comes to generalize and ameliorate the kernel method. Actually, [1] proposed the first local linear estimator model of the regression operator when the explanatory variable takes values in a Hilbert space. When the regressors take values in semi-metric space, [2] introduced another version of the local linear estimator of the regression operator. This last method has been extended to estimate the conditional distribution and its derivatives ([7], [15], [8] and

[14]). When the derivatives estimator of regression provide us about the behaviour of both regression shape and regression mode. Hence, in the sequel, we attempt to give the uniform almost complete convergence of local mode regression.

2. Model Framework and main results

We consider n pairs $(X_i, Y_i)_{i=1,\dots,n}$ identically and independently distributed as (X, Y), this last is valued in $\mathcal{F} \times \mathbb{R}$, where $\mathcal{F}$ is a semi-metric space equipped with a semi-metric d
The mode regression function θ on $\mathcal{F}$ is

$$\theta(x) = \sup_{x \in \mathcal{F}} m(x)$$

whereas, the mode regression estimator $\hat{\theta}$ is defined by

$$\hat{\theta}(x) = \sup_{x \in \mathcal{F}} \hat{m}(x)$$

where : m and $\hat{m}$ are regression and regression estimator respectively. Obviously, mode regression has a relation with regression. So, to get the estimator of local linear regression we minimize the following quantity :

$$\min_{(a,b)\in\mathbb{R}^2} \sum_{i=1}^{n} |Y_i - a - b\beta(X_i, x))|^2 K(h_K^{-1}\delta(X_i, x)) \tag{1}$$

where $\beta(.,.)$ and $\delta(.,.)$ are two functions defined from $\mathcal{F} \times \mathcal{F}$ to $\mathbb{R}$, such that:

$$\forall \xi \in \mathcal{F}, \beta(\xi, \xi) = 0, \text{and} \quad d(.,.) = |\delta(.,.)|.$$

K is Kernel and $h_K = h_{K,n}$ is chosen as a sequence of positive real numbers. our study aims to state the uniform almost complete convergence of $\hat{\theta}$ on some subset $S_{\mathcal{F}}$ of $\mathcal{F}$ such that

$$S_{\mathcal{F}} \subset \cup_{k=1}^{d_n} B(x_k, r_n)$$

where $x_k \in \mathcal{F}$ and r_n (resp d_n) is a sequence of positive real numbers.
We suppose that our estimator satisfies some conditions which are commonly used in many studies of local linear method for functional data, we have :

Theorem 2.1. *(* [12]*)*

$$\sup_{x \in S_{\mathcal{F}}} |\hat{\theta}(x) - \theta(x)| = O(h^b) + O_{a.co.}\left(\sqrt{\frac{\ln d_n}{n\phi_x(h)}}\right)$$

the theoreme's proof can be deduced directly from a Taylor development of $\hat{m}^{(1)}(\hat{\theta}(x))$ around $\theta(x)$ also some technical lemmas.

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$M^X/M/1$ QUEUEING SYSTEM WITH WAITING SERVER, K-VARIANT VACATIONS AND IMPATIENT CUSTOMERS

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Abstract. We consider an $M^X/M/1$ queueing system with waiting server, K-variant vacations, reneging, and retention of reneged customers. We analyze the model using probability generating function (PGF) method. Then, we derive various queueing system characteristics.

2010 Mathematics Subject Classification. 60K25, 68M20, 90B22.

Keywords and phrases. Variant multiple vacations, impatient customers, probability generating function.

1. Introduction

Vacation queueing models with customers impatience have been widely studied due to their large applications in different areas including service systems, communication systems, production and manufacturing systems, and so on (e.g., [3], Ye et al. [4], Bouchentouf and Guendouzi [2]). In this work, we consider an $M^X/M/1$ queueing system at which customers arrive in batches according to a Poisson process with rate λ. Let X denote the batch size random variable of the arrival with probability mass function $P(X = l) = b_l$, $l = 1, 2, ...$ The service time is assumed to be exponentially distributed with parameter μ. The customers are served on FCFS discipline. When the busy period is ended, the server waits a random period before taking a vacation, this waiting time is assumed to be exponentially distributed with parameter η. When duration of the waiting server expires, the server leaves for vacation. Then, at a vacation period termination, if it finds a customer at the vacation completion instant, it comes back to the busy period, otherwise, it takes a finite number, say K, of successive vacations. When the K consecutive vacations are complete, the server returns to busy period and depending on the arrival batch of customers, it stays idle or busy. The period of a vacation follows an exponential distribution with parameter ϕ. During vacation period, each incoming customer starts up an impatience timer independently of the other customers in the system, assumed to be exponentially distributed with parameter ξ. The impatient customers may leave the system with probability α or retained in the system with probability $\alpha' = 1 - \alpha$. The inter-arrival times, batch sizes, waiting server times, vacation times, service time, and impatience times are independent of each other.

2. The equilibrium state distribution

Let $L(t)$ be the number of customers in the system and $S(t)$ denote the status of the server at time t, such that

$$S(t) = \begin{cases} j, & \text{when the server is taking the } (j+1)th \text{ vacation at time } t, \\ & \ j = \overline{0, K-1}; \\ K, & \text{the server is in busy period at time } t. \end{cases}$$

The bi-variate $\{(L(t); S(t)); t \geq 0\}$ represents two dimensional infinite state continuous-time Markov chain with state space $\Omega = \{(n, j) : n \geq 0, j = \overline{0, K}\}$.

Let $P_{n,j} = \lim_{t \to \infty} P\{L(t) = n, S(t) = j\}$, $n \geq 0, j = \overline{0, K}$ denote the system state probabilities of the process $\{(L(t), S(t)), t \geq 0\}$.

Theorem 2.1. *If $\lambda E(X) < \mu$, then the steady-state-probabilities $P_{n,j}$ are given as*

$$P_{.,j} = \sum_{n=0}^{\infty} P_{n,j} = A^{j-1} P_{0,0}, \qquad j = \overline{0, K-1},$$

and

$$P_{.,K} = \sum_{n=0}^{\infty} P_{n,K} = \frac{1}{\mu - \lambda B'(1)} \left\{ \frac{\phi \lambda B'(1)}{\alpha \xi + \phi} \frac{1 - A^K}{A(1-A)} + \frac{\mu \alpha \xi}{\eta C} \right\} P_{0,0},$$

where

$$P_{0,0} = \left\{ \frac{\mu \alpha \xi}{\eta C(\mu - \lambda B'(1))} + \frac{1 - A^K}{A(1-A)} \left(\frac{\phi \lambda B'(1)}{(\mu - \lambda B'(1))(\alpha \xi + \phi)} + 1 \right) \right\}^{-1},$$

such that

$$A = \frac{\phi C}{\alpha \xi},$$

with

$$C = \int_0^1 e^{\frac{\lambda}{\alpha \xi} H(x)} (1-x)^{\frac{\phi}{\alpha \xi} - 1} dx, \quad \text{and} \quad H(z) = \int_0^z \frac{B(x) - 1}{1 - x} dx,$$

where $B(x)$ is the probability generating function of the batch arrival size X, and $B'(1) = E(X)$ is the first moment of random variable X.

3. System Performance measures

– The probability that the server is idle during busy period.

$$P_{0,K} = \frac{\alpha \xi}{\eta C} P_{0,0}.$$

– The probability that the server is in vacation period.

$$P_v = \sum_{j=0}^{K-1} A^{j-1} P_{0,0} = \frac{1 - A^K}{A(1-A)} P_{0,0}.$$

– The probability that the server is serving customers during busy period.

$$P_b = 1 - P_v - P_{0,K}.$$

– The mean system size when the server is on vacation.

$$E[L_V] = \frac{\lambda B'(1)}{\alpha\xi + \phi} \frac{1 - A^K}{A(1-A)} P_{0,0}.$$

– The mean system size when the server is in busy period.

$$E[L_K] = \left[\frac{\phi\lambda B'(1)}{(2\alpha\xi + \phi)(\mu - \lambda B'(1))} + \frac{\phi(2\mu + \lambda B''(1))}{2(\mu - \lambda B'(1))^2} \right] E[L_V]$$
$$+ \frac{\mu\lambda(2B'(1) + B''(1))}{2(\mu - \lambda B'(1))^2} P_{0,K}.$$

– The mean system size.

$$E[L] = E[L_V] + E[L_K].$$

– The mean queue length.

$$E[L_q] = E[L] - \left[1 - \sum_{j=0}^{K} P_{0,j} \right].$$

– The mean number of customers served per unit time.

$$N_s = \mu P_b.$$

– The average rate of reneging.

$$R_a = \alpha\xi E[L_V].$$

– The average rate of retention of impatient customers.

$$R_e = (1 - \alpha)\xi E[L_V].$$

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NON PARAMETRIC ESTIMATION WITH K NEAREST NEIGHBORS METHOD

NADJET BELLATRACH AND WAHIBA BOUABSA

ABSTRACT. It is well known that the nonparametric estimation of the regression function is highly sensitive to the presence of even a small proportion of things that aren't part of the main group in the data. To solve the problem of typical observations when the covariates of the nonparametric component are functional, the robust estimates for the regression parameter and regression operator are introduced.

The main propose of the paper is to think about data-driven methods of selecting the number of neighbors in order to make the proposed processes fully automatic. We use the k Nearest Neighbors procedure (kNN) to construct the kernel estimator of the proposed strong and healthy model. Under some regularity conditions, We mention the results of consistency for kNN functional estimators, for kNN functional estimators, which are uniform in the number of neighbors (UINN). What's more, a simulation study and an empirical application to a real data analysis of octane gasoline predictions are carried out to illustrate the higher predictive performances and the usefulness of the KNN approach.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 62H12, 62G07; Secondary 62G35, 62G20.

KEYWORDS AND PHRASES. Functional data analysis; quantile regression; NN method; uniform nearest neighbor (UNN) consistency; functional nonparametric statistics; almost complete convergence rate.

1. ESTIMATION MODEL

In this article, the purpose is to evaluate the impact of the functional variable X on the real variable Y using the robust estimation of the regression function

Let us introduce n pairs of random variables $(X_i; Y_i)_{i\geq 1}$, that we assume drawn from the pair (X, Y),, which is valued in $\mathcal{F} \times R$, where $\mathcal{F}$ is a semi-metric space equipped with a semi-metric d.The relationship between X and Y is given by $Y = r(X) + \epsilon$, where ϵ represents an independent random variable of X with a symmetric distribution. The robust methode was defined, for any loss function $\rho(.,.)$ on IR, as the unique minimizer with respect to (w.r.t.) the component t in the model $\Gamma_x(t) = \mathbb{E}\left[\rho(Y, t)/X = x\right]$. The theoretical estimator of this model is defined by

$$\theta_x = \arg\min_{t\in\mathbb{R}} \Gamma_x(t) \tag{1}$$

1

According to Eq. (1), the best approximation of Y given X is based on the estimation of θ_x denoted by $\widehat{\theta}_x$, given by $\widehat{\theta}_x = \arg\min_{t\epsilon\mathbb{R}} \widehat{\Gamma}_x(t)$ where:

$$\widehat{\Gamma}_x(t) = \frac{\sum_{i=1}^n k\left(h_k^{-1} d(x, X_i)\right) \rho(Y_i, t)}{\sum_{i=1}^n k\left(h_k^{-1} d(x, X_i)\right)} \tag{2}$$

with k is a kernel function and $h_k = \min\left\{h \in \mathbb{R}^+ \ such\ that\ \sum_{i=1}^n \mathbb{I}_{B(x,h)}(X_i) = k\right\}$ with is given as a sequence of integers.

1.1. **Main results.** To establish the almost complete convergence of $\widehat{\theta}_x$ uniformly in the numbers of neighbors $k \in (k_{1,n}, k_{2,n})$, we need the following conditions and notations:

(A1) for all $r > 0, \mathbb{P}(X \in B(x,r)) = \phi_x(r) > 0$ such that, for all $s \in (0,1), \lim_{r\to 0} \frac{\phi_x(sr)}{\phi_x(r)} = \tau_x(s) < \infty$

(A2) The function Γ is such that:

(i) The function $\Gamma_x(.)$ is of class C^2 on $[\theta_x - \delta, \theta_x + \delta], \delta > 0$

(ii) $\forall t \in [\theta_x - \delta, \theta_x + \delta], \forall (x_1, x_2) \in \mathcal{N}_x \times \mathcal{N}_x, |\Gamma_{x_1}(t) - \Gamma_{x_2}(t)| \leq C d^b(x_1, x_2)$, where $\mathcal{N}_x$ is a fixed neighborhood of x.

(iii) For each fixed $t \in [\theta_x - \delta, \theta_x + \delta]$ the function $\Gamma_.(t)$ is continuous at x.

(A3) The function ρ is a strictly convex, continuous and differentiable w.r.t. the variable t, and its derivative, $\psi(y,t) = \frac{\partial \rho(y,t)}{\partial t}$, fulfills $\mathbb{E}\left[|\psi(y,t)|^2\right] < C < \infty$ and $\mathbb{E}\left[|\psi(y,t)|^p\right] < C < \infty, p > 1$.

(A4) The kernel function k is supported within $(0, 1/2)$ and the derivative function of k is continuous on $(0, 1/2)$ such that
$0 < C\mathbb{I}_{\frac{0,1}{2}}(.) \leq k(.) \leq C^{'\mathbb{I}_{\frac{0,1}{2}}}(.)$ and $k(1/2) - \int_0^{\frac{1}{2}} k'(s)ds > 0$
where $\mathbb{I}_A$ denotes the indicator function of the set A.

(A5) Let define the class κ of functions by $\kappa = \left\{. \mapsto k(\gamma^{-1} d(x,.)), \gamma > 0\right\}$ which is a pointwise measurable $class^4$ such that $\sup_Q \int_0^1 \sqrt{1 + \log \mathcal{N}(\epsilon \|F\|_{Q,2})} d\epsilon < \infty$, where the supremum is taken over all probability measures Q on the space $\mathcal{F}$ with $Q(\mathbb{F}^2) < \infty$ and $\mathcal{F}$ is the envelope function5 of the set $\mathbb{K}$.

(A6) The sequence of numbers $(k_{1,n})$ verifies
$\phi_x^{-1}\left(\frac{k_{2,n}}{n}\right) \to 0$ and $\frac{logn}{min(n\phi_x^{-1}\left(\frac{k_{2,n}}{n}\right), k_{1,n})}$

C or/and C' denotes a generic positive constant. In the following theorem, we present the consistency result.

Theorem 1.1. *Assume that conditions (A1)-(A6) are satisfied, then* $theta_x$ *exists and is unique almost surely for all larger value of* n *. Furthermore, if* $\Gamma_x''(\theta_x) \neq 0$*, we have*

$$\sup_{k_{1,n} \leq k \leq k_{2,n}} \left|\widehat{\theta_x} - \theta_x\right| = O\left(\phi_x^{-1}\left(\frac{k_{2,n}}{n}\right)^{min(k_1,k_2)}\right) + O_{a.co}\left(\sqrt{\frac{log(n)}{k_{1,n}}}\right).$$

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NONPARAMETRIC CONDITIONAL DENSITY FUNCTION ESTIMATION FOR RANDOMLY CENSORED DATA

IMANE BOUAZZA AND FATIMA BENZIADI

ABSTRACT. We employ in this paper a nonparametric estimator of the conditional density function in the framework of independent as well as $\alpha-$mixing data case when the variable of interest is subject to right-censorship, by using both the classical and recursive methods. [5] and [6] establish first the uniform strong consistency rates on a compact and compare the mean squared errors of theses two kernel estimators. Some simulations are carried out to confirm that the resulting recursive estimator performs better than the non-recursive ones.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 62G05, 62G07, 62G08, 62G20, 62H12.

KEYWORDS AND PHRASES. Conditional density function, Kernel estimator, Recursive kernel estimation, Censored data, Kaplan-Meier estimator, Uniform almost sure convergence.

1. INTRODUCTION

Much recent work has been carried out for the nonparametric estimation of the conditional models in the context of censored data, both in a theoretical framework and application, let us cite, among many others [4], [5], [6] and [8]. These models play a crucial role in nonparametric prediction. This is what make the method of studying the estimators expanded and many statisticians tend to use the recursive methods because of its many benefits. The computational advantage of recursive estimators on their non-recursive versions is obvious: their update, from a sample of size n to one of size $n+1$, can be computed instantly and does not require extensive storage of data.

Thus, to make the paper more organized and clear, we will present first some background of censored data. Let $T_1, T_2, \ldots, T_n$ be strictly stationary, non-negative survival times with continuous distribution function F admitting a density f. Then, assuming that $C_1, C_2, \ldots, C_n$ is a sequence of i.i.d. right censoring times, and let G denote the unknown cumulative distribution function of C, which is estimated by [3] and defined as follows

$$\overline{G}_n(t) = 1 - G_n(t) = \begin{cases} 0 & ; t \geq Y_{(n)} \\ \displaystyle\prod_{k=1}^{n} \left(1 - \frac{1-\Delta_{(k)}}{n-k+1}\right)^{\mathbb{I}\left(Y_{(k)} \leq t\right)} & ; t < Y_{(n)} \end{cases}$$

where $Y_{(1)} < Y_{(2)} < \cdots < Y_{(n)}$ are the order statistics of $(Y_k)_{k \in \{1,\ldots,n\}}$ and $\Delta_{(k)}$ is the corresponding concomitant of $Y_{(k)}$.

We will also define respectively the support right endpoints τ_F and τ_G of the survival functions $\overline{F}$ and $\overline{G}$ as

$$\tau_F := \sup\{t \in \mathbb{R} : \overline{F}(t) > 0\} < \infty \ \ and \ \ \tau_G := \sup\{t \in \mathbb{R} : \overline{G}(t) > 0\}$$

and such that $\tau_F < \tau_G$ with $\overline{G}(\tau_F) > 0$.

2. Conditional Density Estimators

Consider in the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, n pairs of random variables (X_k, T_k) that we assume drawn from the pair (X, T) which is valued in $\mathbb{R}^d \times \mathbb{R}$. The problem of this paper is to propose a nonparametric estimate of the conditional probability density function (c.p.d.f.) involved of Y given $X = x$ when the response variables Y_k are rightly censored. The variables $\{(X_k, T_k), k \geq 1\}$ and $\{C_k, k \geq 1\}$ are assumed to be independent. Thus, in the right censoring model, the observed data are the triplet $(X_k, Y_k, \Delta_k), k = 1, \ldots, n$, with

$$Y_k = \min\{T_k, C_k\} \ and \ \Delta_k = \mathbb{I}_{[T_k \leq C_k]}, \quad 1 \leq k \leq n,$$

where $\mathbb{I}_A$ denotes the indicator function of the set A. For $x \in \mathbb{R}^d$, we supposed that the conditional probability distribution of T given $X = x$ exists and given by

$$\forall t \in \mathbb{R}, \quad F(t/x) = \mathbb{P}[T \leq t/X = x].$$

Now, we are in position to define in both cases the estimators of the desired model based on the randomly right-censorship, such that for all $x \in \mathbb{R}^d$

$$\phi^x(t) = \frac{\partial F}{\partial t}(t/x)$$

2.1. **Classical Case.**

Classically, the conditional density function $\phi^x(t)$ is estimated by [4] where the step kernel is replaced by a smooth pdf L. Here, $(X_n, T_n)_{n\geq 1}$ is a stationary α-mixing sequence of rvs, with coefficient $\alpha(n)$ which satisfies for some $\nu > d + 4$, $\alpha(n) = O(n^{-\nu})$. Thus,

$$\widetilde{\phi}_n^x(t) = \frac{\sum_{k=1}^{n} \Delta_k \overline{G}_n^{-1}(Y_k) K\left(\frac{x - X_k}{h_{n,K}}\right) L\left(\frac{t - Y_k}{h_{n,L}}\right)}{h_{n,L} \sum_{k=1}^{n} K\left(\frac{x - X_k}{h_{n,K}}\right)} := \frac{\widetilde{g}_n(x,t)}{\gamma_n(x)}$$

where

$$\widetilde{g}_n(x,t) := \frac{1}{n h_{n,K}^d h_{n,L}} \sum_{k=1}^{n} \Delta_k \overline{G}_n^{-1}(Y_k) K\left(\frac{x - X_k}{h_{n,K}}\right) L\left(\frac{t - Y_k}{h_{n,L}}\right).$$

With

- $g(\cdot,\cdot)$ the joint probability density function assumed to be bounded and continuously differentiable up to order 3 and $\gamma(\cdot)$ the marginal density one assumed to be twice continuously differentiable;
- The functions K and L are kernels assumed to be Lipschitzian in this paper;

- h_n is a sequence of positive real numbers tending to 0 as $n \to \infty$ and satisfies

$$\frac{n h_n^{(\nu+d+4)(d+1)/(\nu-d-4)}}{\log^{(\nu+1)/(\nu-d-4)} n \log_2^{6/(\nu-d-4)} n} \to \infty \ as \ \ n \to \infty.$$

Throughout the rest of the paper, let $\mathcal{C}$ and Ω be two compact sets of $\mathbb{R}^d$ and $\mathbb{R}$ respectively.

Theorem 2.1. [5] *Under certain classical assumptions on the bandwidth and regularity conditions on the kernels, the joint and marginal densities, we will have*
(1)

$$\sup_{x\in\mathcal{C}} \sup_{t\in\Omega} \left| \widetilde{\phi}_n^x(t) - \phi^x(t) \right| = O\left\{ \max\left(\left(\frac{\log n}{n h_n^{d+1}} \right)^{1/2}, h_n^2 \right) \right\} \ a.s. \ \ as \ \ n \longrightarrow \infty.$$

Now, we are in position to give the second estimate

2.2. **Recursive Case.** The recursive version of the previous kernel estimator is proposed by [6] and defined for $(x, y) \in \mathbb{R}^d \times \mathbb{R}$ as

$$\widehat{\phi}_n^x(t) = \frac{\sum_{k=1}^n h_k^{-(d+1)} \Delta_k \overline{G}_n^{-1}(Y_k) K\left(\frac{x - X_k}{h_k}\right) L\left(\frac{t - Y_k}{h_k}\right)}{\sum_{k=1}^n h_k^{-d} K\left(\frac{x - X_k}{h_k}\right)} := \frac{\widehat{g}_n(x,t)}{\gamma_n(x)}$$

where

$$\widehat{g}_n(x,t) := \frac{1}{n} \sum_{k=1}^n \frac{1}{h_k^{d+1}} \Delta_k \overline{G}_n^{-1}(Y_k) K\left(\frac{x - X_k}{h_k}\right) L\left(\frac{t - Y_k}{h_k}\right);$$

and

$$\gamma_n(x) := \frac{1}{n} \sum_{k=1}^n \frac{1}{h_k^d} K\left(\frac{x - X_k}{h_k}\right).$$

Note that, the following result is obtained when $(X_n, T_n)_{n\geq 1}$ is a stationary independent and identically distributed sequence of rvs.

Theorem 2.2. [6] *Under standard conditions on regularity of functions and let* $h_n^- = \inf_{k=1,\ldots,n} h_k$ *and* $h_n^+ = \sup_{k=1,\ldots,n} h_k$. *We have*

$$(2) \qquad \sup_{x\in\mathcal{C}} \sup_{t\in\Omega} \left| \widehat{\phi}_n^x(t) - \phi^x(t) \right| = O\left\{ \max\left(\left(\frac{\log n}{n h_n^{-(d+1)}} \right)^{1/2}, h_n^{+2} \right) \right\} \ a.s.$$

3. Simulation Study

To compare the finite-sample performance of both methods (the recursive and the classical kernel ones), we are in position to consider the following model: $Y = r(X) + \epsilon$ with $r(X) = \exp(X - 0.2)$ (for more examples of models, the reader can refer to the original articles [5] and [6]), where the random variables X and ϵ are i.i.d. and follow respectively the normal distribution $\mathcal{N}(0, 1)$ and $\mathcal{N}(0, \sigma)$. Thus, [6] choose three censoring type of

τ=(10, 40, 70) in order to control the effect of this factor in the efficiency of both estimators, by fixing the sample size $n = 200$ for each case. Then, the MSE results under the randomly right-censorship are given in the table below

τ	MSE(KERNEL)	MSE(RECURSIVE)
10	0.79	0.29
40	1.32	1.18
70	2.17	2.65

Censored Data MSE-Results.

4. Conclusion

In this summary, we provided two type of estimators of the conditional density function. We considered the case where the data are i.i.d. and $\alpha-$mixing, we dealt with the strong almost sure convergence of these estimators, as well as a simple numerical study. On one hand, there is no big difference in the calculations and the assumptions except for certain conditions concerning the bandwidths h and the inequalities adapted to each case. The necessary proofs and discussion of these results are detailed in the papers of [5] and [6].

On the other hand, a simulation comparison between the two estimators showed clearly that the recursive method is slightly better than the classical kernel one. However, the recursive estimator is strongly affected by the presence of a high censoring rate and to be more objective we deal here only with the i.i.d. case, but the same still true for the other case (α-mixing).

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NONPARAMETRIC RELATIVE ERROR ESTIMATION OF THE REGRESSION FUNCTION FOR TWICE CENSORED DATA AND UNDER $\alpha-$ MIXING CONDITION.

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ABSTRACT. In this paper, we are interested in the nonparametric estimation of the regression function for twice censored data under strong mixing condition. First, we built a kernel estimator for the error relative regression function when the r.v. of interest is twice censored and satisfies the $\alpha-$ mixing property. Then, we study its asymptotic behavior.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 62G05, 62G20.

KEYWORDS AND PHRASES. Kernel estimate, Patilea-Rolin estimator, Relative regression error, Strong mixing condition, Twice-censored data, Uniform almost sure consistency.

1. DEFINE THE PROBLEM

Consider the regression model $Y = r(X)+\epsilon$, where Y is a random variable (rv) of interest, X is a random covariate such that the error ϵ has zero mean and is uncorrelated with X. In the classical regression model, the purpose is to find a function $r^*(x)$ which achieves the minimum of the mean squared error

$$\mathbb{E}\left[(Y - r(x))^2 \,|X\right], \tag{1}$$

however, this loss function which is considered as a measure of the prediction performance may be unadapted to some situations. So, in the following, we circumvent the limitations of this classical regression by estimating the operator of regression with respect to the minimization of the following mean squared relative error (MSRE), for $Y > 0$

$$\min_r \mathbb{E}\left[\left(\frac{(Y - r(x))}{Y}\right)^2 |X\right]. \tag{2}$$

In many practical applications, it happens that one is not able to observe a subject's entire r.v. of interest. The most current forms of such incomplete data are censorship and truncation. [1] and [2] studied the consistency and asymptotic normality of the kernel estimator of the classical regression function for right censored data in the i.i.d. case and strong mixing case respectivly.
The model studied here in the twice censored introduced by [6]. The classical regression for this model was considered by [5] and [3]. The relative error regression has been studied by [4].
Our goal is to built a new kernel estimator of the mean squared relative error prediction for the regression function under twice censored for $\alpha-$ mixing

data. Then, we establish the uniform almost sure consistency. Our results are illustrated by some simulations.

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NON-PARAMETRIC DENSITY ESTIMATION FOR POSITIVE AND CENSORED DATA: APPLICATION TO LOG-NORMAL KERNEL.

SARAH GHETTAB AND ZOHRA GUESSOUM

Abstract. The density function is an important problem for inference with censored data. We propose a new type of kernel estimator for density function that performs well at the boundary, when the variable of interest is positive (e.g. lifetime) and right censored. Following Chaubey [?] who introduced and studied a new type of asymmetric kernels density estimator, for complete data. We give the rate of almost sure uniform convergence of this estimator for censored data. We compare by simulation the performance of the Log-normal kernel density estimator , with the symmetric kernel, performed by Blum and Susarla (1980)[?].

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Keywords and phrases. Asymmetric kernel, Censored data , Strong uniform consistency.

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Normalité asymptotique de l'estimateur du coefficients d'un AR[1] sous dependance faible

ZEMOUL S.I. (1),BERKOUN Y. (2)

Résumé :
On s'intéresse quelques propriètés asymptotiques de l'estimateur des moindres carrés du paramètre d'un processus autoregressif d'ordre un (AR(1)) lorsque les innovations sont faiblement dépendantes dans un certain sens. Le résultat est basé sur certains théorèmes relatifs aux variables négativement associées (NA) et faiblement dépendantes.

2010 Mathematics Subject Classification. Primary 60G70,Secondary 60G10.

Mots-clès :Variables associées, processus linéaire, dépendance faible, modèle autoregressif, estimateur des moindres carrés.

Introduction
Soit $(X_t)_t$ un processus autoregressif d'ordre un défini par :

$$X_t = \rho X_{t-1} + \epsilon_t, t = 1, 2, \ldots,\ 0 < |\rho| \leq 1, \tag{1.1}$$

où $(\epsilon_t)_t$ est une suite de variables aléatoires négativement associées indépendantes de X_0 et qui sont négativement associées (NA). Notons par $\widehat{\rho}$, l'estimateur des moindres carrés ordinaires de ρ donné par

$$\widehat{\rho} = \frac{\sum_{t=1}^{n} X_t X_{t-1}}{\sum_{t=1}^{n} X_{t-1}^2}, \tag{1.2}$$

On s'intéresse aux propriétès asymptotiques de $\widehat{\rho}$ lorsque les innovations sont NA. Sous certaines conditions que l'on spécifiera dans la suite, la normalité asymptotique et la consistance sont obtenues.

1-Définitions et résultats préliminaires

Dans cette section, on donne certaines définitions et quelques résultas qui vont nous servir pour démontrer nos résultats.

Définition 1.1 Une famille des variables aléatoires $(X_1, \ldots, X_n)$ est dite négativement associée (NA), si pour tous sous-ensembles disjoints A_1 et A_2 de $(1, ..., n)$ on a

$$Cov(f_1(X_i, i \in A_1), f_2(X_j, j \in A_2)) \leq 0$$

où f_1 et f_2 sont deux fonctions croissantes par rapport à chaque composante. Une suite $(X_n)_n$ de variables aléatoires est dite NA si toute sous-famille finie de variables aléatoires est NA.

Définition 1.2(voir [1])

Un processus $(X_t)_t$ est dit $(\theta, \pounds, \psi)$- faiblement dependent s'il existe une suite $(\theta_r)_r$, $r \in \mathbb{N}$ décroissante vers 0 quand r tend vers l'infini et une fonction ψ définie sur $\pounds_n \times \pounds_m \times \mathbb{N}^2$ tels que pour $(h,k) \in \pounds_u \times \pounds_v$ et $(u,v) \in \mathbb{N}^2$

$$Cov(g(X_{i_1},...,X_{i_n}), h(X_{j_1},...,X_{j_m})) \leq \psi(g,h,n,m)\theta_r$$

Pour tout $(i_1,...,i_n)$ et $(j_1,...,j_m)$ avec $i_1 < .... < i_n \leq i_n + r \leq j_1 < .... < j_m$.

$\pounds_n$ est la classe des fonctions Lipschitziennes réelles, bornées par 1 et définies sur $\mathbb{R}^n$ $(n \in \mathbb{N}^*)$ et $\pounds = \bigcup_{n=1}^{\infty} \pounds_n$. i.e.,

$$\pounds_n = (g \in \mathbb{L}^{\infty}(\mathbb{R}^n)\ Lip(g) < \infty,\ \|g\|_{\infty} \leq 1)$$

$Lipf$ représente le module de continuité de Lipschitz de f défini par

$$Lip(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|_1}$$

avec $\|x - y\|_1 = \sum_{i=1}^{n} |x_i - y_i|$.

Pour diverses spécifications de la fonction ψ, on obtient les dépendance suivantes
- κ-dépendance avec $\psi(n,m,a,b) = nmLip(g)Lip(h)$ notée $\kappa(r)$
- θ-dépendance avec $\psi(n,m,a,b) = mb$
- η-dépendance avec $\psi(n,m,a,b) = nLip(g) + mLip(h)$
-λ-dépendance avec $\psi(n,m,a,b) = nmLip(p)Lip(g) + nLip(g) + mLip(h)$
-ζ-dépendance avec $\psi(n,m,a,b) = min(n,m)Lip(g)Lip(h)$

- **Exemples**
- Si $(X_n)_n$ est une suite de variables aléatoires associeés et centrées, alors $(X_n)_n$ est ζ-faiblement dépendant avec $\theta_r = sup_i \sum_{j:|i-j| \geq r} Cov(X_i, X_j)$

Les processus linéaires avec des innovations indépendantes sont sous certaines conditions η-faiblement dépendants (voir [2]).Le lemme suivant nous donne la dépendance faible d'un processus linéaire sous innovations associées.

Lemme 1. *(voir [3]) Soit $X_t = \sum_{i \geq 0} c_i \varepsilon_{t-i}$ un processus linéaire où la suite $(\varepsilon_t)_t$ est formée de variables aléatoires associées. On suppose que $E(|\varepsilon_t|^{2+\delta}) < \infty$, $\delta > 0$ et que $\Sigma_{j \geq 1} |c_j| < \infty$. Alors, le processus $(X_t)_t$ est ζ-faiblement dépendant.*

Lemme 2. *Soit $(X_n)_n$ une suite de variables aléatoires NA telles que $E(|X_t|^{\beta}) < \infty$ pour $1 \leq \beta < 2$, alors*

$$\frac{1}{n^{\frac{1}{\beta}}} \sum_{k=1}^{n} (X_k - E(X_k)) \xrightarrow{p.s} 0$$

Lemme 3. *Soit* $(X_n)_n$ *une suite de variables aléatoires NA telles que* $E(X_n) = 0, \forall n$ *et* $E(|X_n|^{pq}) < \infty$ *pour* $1 \leq P \leq 2$, $q \geq 1$.*Posons* $S_n = \sum_{i=1}^{n} X_i$ *et notons* $(a_n)_n$ *une suite positive de réels croissante vers l'infini telle que*

$$\sum_{n=1}^{\infty} a_n^{-p}(a_n^p - a_{n-1}^p)^{-q+1} E(|X_n|^{pq}) < \infty$$

alors $\frac{S_n}{n} \xrightarrow{p.s} 0$

II- Résultas

Remarquons d'abord que

$$\widehat{\rho} = \frac{\sum_{t=1}^{n} X_t X_{t-1}/n}{\sum_{t=1}^{n} X_{t-1}^2/n} = \frac{U_n}{V_n} \tag{2.1}$$

Pour montrer la normalité asymptotique, il suffit de montrer que U_n converge en loi et que V_n converge en probabilité et d'utiliser le théorème de Slutsky.

2.1 Consitance de $\widehat{\rho}$ sous NA

Théorème 1. *(voir [4]) Soit* $(X_t)_t$ *defini par (1.1) et on suppose que le processus des innovations* $(\epsilon_t)_t$ *est faiblement stationnaire vérifiant*

$$E(\epsilon_n) = 0,\ E(\epsilon_n^2) = \sigma_1^2 < \infty,\ 0 < \sigma_1^2 + 2\sum_{i=1}^{+\infty} E(\epsilon_1\epsilon_{1+i}) < \infty,\ sup_n E(|\epsilon_n|^{2+\delta}) < \infty,\ pour\ \delta > 0 \tag{2.2}$$

Pour n tendant vers l'infini, on a

a) $\frac{1}{n}\sum_{i=1}^{n} \epsilon_i^2 \xrightarrow{p.s} \sigma_1^2$

b) $\frac{1}{n}\sum_{i=1}^{n} \epsilon_i X_{i-1} \xrightarrow{p.s} \sum_{i=1}^{+\infty} \rho^i E(\epsilon_1\epsilon_{1+i})$

c) $\frac{1}{n}\sum_{i=1}^{n} X_{i-1}^2 \xrightarrow{p.s} \frac{1}{1-\rho^2}(\rho^2 + 2\sum_{i=1}^{+\infty} \rho^i E(\epsilon_1\epsilon_{1+i}))$

d)

$$\widehat{\rho} = \frac{\sum_{t=1}^{n} X_t X_{t-1}}{\sum_{t=1}^{n} X_{t-1}^2} \xrightarrow{p.s} \rho + \frac{(1-\rho^2)\sum_{i=1}^{+\infty} \rho^{i-1} E(\epsilon_1\epsilon_{1+i})}{\sigma_1^2 + 2\sum_{i=1}^{+\infty} \rho^i E(\epsilon_1\epsilon_{1+i})}$$

Remarques

1- Il en découle de ce qui précède, que pour $0 < \rho < 1$, l'estimateur $\widehat{\rho} \leq \rho$ et donc n'est pas forcément consistent. La consistence est oblenue que ssi $E(\epsilon_1\epsilon_{1+i}) = 0\ \forall i \geq 1$

2- Si le processus $(\epsilon_t)_t$ est strictement stationnaire, la condition $sup_n E(|\epsilon_n|^{2+\delta}) < \infty$ n'est pas nécessaire.

2.2 Normalité asymptotique de $\widehat{\rho}$ sous NA

Pour obtenir la normalité asymptotique de $\widehat{\rho}$ sous innovations associées, nous avons besoin des hypothèses suivantes.

- **H1** : $E(|\epsilon_n|^{2(q+\delta)}) < \infty,\ q > 2$, et que les hypothèses du lemme 1 sont satisfaites.

- **H2** : $\theta_r = O(r^{-m}),\ m > \frac{q(q+\delta)}{2\delta},\ \Sigma_r r^\delta \theta_r^{1-s/2} < \infty,\ s > 2$

Posons $X_{i,h} = (X_{i-h}, \ldots, X_i)'$, $\Gamma_X(h) = E(X_0 X_{h,h})$, $\Sigma_X = (\Sigma_j |c_j|)^4 \Sigma_\epsilon$

Σ_ϵ est la matrice d'ordre $(h+l, k+1)$ où l'élément

$$\sigma_{l+1,k+1} = lim_n Cov(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \epsilon_i \epsilon_{i+l}, \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \epsilon_j \epsilon_{j+k}) = \sum_{i=-\infty}^{+\infty} Cov(\epsilon_0 \epsilon_l, \epsilon_i \epsilon_{i+k})$$

Théorème 2. *Sous les hypothèses H1 et H2, alors*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i X_{i+h,h} - \Gamma_X(h) \xrightarrow{d} N(0, \Sigma_X)$$

Remarque

La normalité asymptotique de $\widehat{\rho}$ se déduit du théorème 2 (voir[5]) et du c du théorème 1.

Lemme 4. *(voir [3])*

Soit $(Y_i)_{i\in\mathbb{Z}}$ un processus stationnaire et $H : \mathbb{R}^{(\mathbb{Z})} \to \mathbb{R}$ satisfait la condition suivante :
Notons par $\mathbb{R}^{(\mathbb{Z})} = \bigcup_{I>0} \{z \in \mathbb{R}^{(\mathbb{Z})}\} / z_i = 0, |i| > I\}$, l'ensemble finis des suites de nombres réels .On considère $H : \mathbb{R}^{(\mathbb{Z})} \longrightarrow \mathbb{R}$ tel que si $x, y \in \mathbb{R}^{(\mathbb{Z})}$ coïncident pour tous les indices sauf un , soit disant $s \in \mathbb{Z}$,alors

$$|H(x) - H(y)| \leqslant b_s((\| z \|)^l \vee 1)|x_s - y_s|$$

ou $z \in \mathbb{R}^{(\mathbb{Z})}$ est définit par $z_s = 0$ et $z_i = x_i = y_i$ si $i \neq s$ .Ici $\| x \| = \sup_{i\in\mathbb{Z}} |x_i|$ pour certains $l > 0$ et certaines $b_j \geq 0$ tel que $\sum_j |j| b_j < \infty$.

Supposons qu'une paire de nombres réels (m, m') avec $E|Y_0|^{m'} < \infty$ tel que $m \succeq 1$ et $m' \geq (l+1)m$.Alors :

- *le processus $X_n = H(Y_{n-i}, i \in \mathbb{Z})$ est bien définit dans L^m : c'est un processus strictement stationnaire*
- *Si le processus d'entrée $(Y_i, i \in \mathbb{Z})$ est λ-faiblement dépendent (les coefficients de dépendance faible sont notés par $\lambda_y(r)$), alors X_n est faiblement dépendants et il existe une constante $c > 0$ tel que*

$$\lambda(k) = c \inf_{r\leq[k/2]} \left[\sum_{|j|\geq r} |j|b_j + (2r+1)^2 \lambda_Y(k-2r)^{\frac{m'-1-l}{m'-1+l}} \right]$$

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ON FRACTIONAL AUTOREGRESSIVE MODEL OF ORDER 1 WITH A PERIODIC COEFFICIENT

NESRINE BENAKLEF [(1)] KARIMA BELAIDE [(2)]

Abstract. This paper deals with the effect of introducing a periodic coefficient on FAR(1) model, we present some probabilistic properties and in order to come out with the final conclusions on the autocovariance function's behaviour, we consider a simulation study.
2010 Subject Classification.91B70
Keywords and phrases. Periodic coefficient, periodically correlated, periodic functions, short memory, long memory.

Define the problem

This work consist of introducing a periodic coefficient in the FAR (1) defined by the following form

$$(1 - aL)^d X_t = \varepsilon_t$$

with

- L is lag operator.
- $|a| \leq 1$
- $\varepsilon_t; t \in \mathbb{Z}$ is a white noise, sequence of independent random variable identically distributed with zero mean and finite variance.
- d is an unknown parameter not necessary integer.

This model was introduced by Gonçalves (1987) and subsequently studied by Serroukh (1996). We tackle the question related to the study and the modelling of cyclic phenomena, the model of interest is given as follow

$$(1 - a_t L)^d X_t = \varepsilon_t$$

where a_t it is assumed to be periodic with period $p \in \mathbb{N}$; for all $t \in \mathbb{Z}, \exists i = \{0, ..., p-1\}, m \in \mathbb{Z}$. And we preserve all the same conditions on the other parameters
We show that our model is causal and invertible under a sufficient condition.
The present topic is interested in a certain probabilistic properties and in the asymptotic behaviour of the periodic auto-covariance functions. In the end, we compared the results found with those of the original model through a simulation study.

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ON FRACTIONAL AUTOREGRESSIVE PROCESS OF ORDER 1 WITH STRONG MIXING ERRORS.

DJILLALI SEBA AND KARIMA BELAIDE

Abstract. This work is devoted to the study of the behavior of fractional autoregressive model with strong mixing errors, we establish probabilistic properties: auto-covariance and auto-correlation and its asymptotic behavior under geometrical $\alpha-$mixing assumption, In order to measure the performance of the theoretical results we conduct a simulation study.

2010 Mathematics Subject Classification. 91B70.

Keywords and phrases. Long memory, fractional autoregressive process, dependence, $\alpha-$mixing, autocorrelation.

1. Define the problem

In our work we consider the following model:

$$(1-aL)^d X_t = \varepsilon_t \tag{1}$$

L is lag operator, d, a are constants, ε_t are assumed to be strong mixing.
our model is invertible when:

$$|a| < 1 \quad or \quad |a| = \pm 1 \quad |d| < \frac{1}{2}$$

The model 1 was introduced by Gonçalves (1987)[1] and studied by Serroukh (1996) [4] in the case of independent errors.
In the present work we generalized these results when the errors are $\alpha-$mixing; we deal with geometrical case, therefore we use Cramer conditions in calculus.
In our study we treat the probabilistic behavior of the auto-covariance, autocorrelation functions and their approximation formulas, it exhibits the effect of the parameters d,a and the strong mixing errors on the decay of the functions which allows us to know if the model is long memory or short memory process.
Due to the behavior of the autocorrelation function we have two cases.

- When a is close to 1 the autocorrelation function decreases hyperbolically.
- When a is distant then 1, close to 0 the the autocorrelation function has a fast decay.

We can also say that the strong mixing assumption affect the behavior of the autocorrelation, and when d is bigger the decay is slower.

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ON GENERALIZED QUASI LINDLEY DISTRIBUTION: GOODNESS OF FIT TESTS

SIDAHMED BENCHIHA AND AMER IBRAHIM AL-OMARI

Abstract. In this article, several goodness of fit tests for the generalized quasi Lindley distribution are suggested based on the commonly used simple random sampling (SRS) and ranked set sampling (RSS) methods. These tests includes Kolmogorov-Smirnov test, Anderson-Darling test, Cramer-von Mises test, and Zhang test. The power of the tests and the critical values are obtained based on SRS and RSS schemas for various alternatives. A comparison study is performed to study the goodness of fit tests based on RSS relative to its counterparts in SRS based on the same number of measured units. An application of real data set is given for illustration. The results indicate that the RSS tests performs well as compared to the SRS.

2010 Mathematics Subject Classification. xxxx, xxxx, xxxx.

Keywords and phrases. goodness of fit tests, maximum likelihood estimation, power of test, ranked set sampling, significance level.

1. Define the problem

Ranked Set Sampling (RSS) method was introduced by McIntyre (1952) as an alternative method for collecting data to Simple Random Sampling (SRS). It was proposed to improving precision in estimation of a population mean. The RSS strategy can be described as follow:

(1) select a simple random sample of size h^2 from the desired population. Randomly partition it into h sets of each size h. h is named the set size.
(2) order each set of size h from smallest to largest.
(3) Obtain the i^{th} order statistic from the i^{th} set. (for $i = 1, 2, ..., h$).
(4) Repeat the steps (1)–(3), r times (cycles), to get a ranked set sample of size $M = rm$.

The resulted sample is denoted as $\{X_{[i]j}, i = 1, \dots, h; j = 1, 2, \dots, r\}$ where $X_{[i]j}$ is the i^{th} largest unit in a set of size h in the j^{th} cycle.
A new RSS design was proposed such as double ranked set by Al-Saleh and Al-Kadiri (2000), median ranked set by Muttlak (1997), neoteric ranked set sampling by Zamanzade and Al-Omari (2016), extreme ranked set sampling by Samawi et al (1996), L ranked set by Al-Nasser (2007). For more about RSS, you can see Al-Omari and Bouza (2014), Al-Hadhrami and Al-Omari (2014), Haq et al (2015), Santiago et al (2016), Haq et al (2016). In the literature, some authors apply the goodness of fit based on entropy and empirical distribution function using RSS design such as Al-Omari and Haq (2012) for the inverse Gaussian distribution, Al-Omari and Zamanzade

(2016) for Rayleigh distribution, Al-Omari and Haq (2016) tests for the inverse Gaussian and Laplace distributions using pair ranked set sampling, Al-Omari and Zamanzade (2017) for Laplace distribution. For more details on goodness of fit and entropy you can see Al-Omari (2014), Al-Omari (2015), Al-Omari (2016),Al-Omari and Zamanzade (2018),Ebrahimi et al (1994),Van Es (1992). in this work, we give some goodness of fit test using SRS and RSS design for Generalized Quasi Lindley distribution (GQLD) proposed by Benchiha and Al-Omari(2020).

2. Generalized Quasi Lindley distribution

Generalized Quasi Lindley distribution (GQLD) is propose by Benchiha. S and Al-Omari. A (2020). The GQLD is a sum of two independent quasi Lindley distributed random variables. Let X follows a GQLD with parameters θ and ξ then the probability density function (pdf) of X is given by

(1)
$$g_{GQLD}(x;\psi,\xi)=\frac{\xi^2\left(\frac{\psi^2x^3}{6}+\xi\psi x^2+\xi^2x\right)\mathrm{e}^{-\psi x}}{(\xi+1)^2};\ x\geq 0,\ \xi>-1,\ \psi\geq 0.$$

and its cumulative distribution function is defined as:

(2)
$$G_{GQLD}(x;\psi,\xi\ )=1-\frac{\left(\psi^3x^3+3\left(2\xi+1\right)\psi^2x^2+6\left(\xi+1\right)^2\left(\psi x+1\right)\right)\mathrm{e}^{-\psi x}}{6\left(\xi+1\right)^2}.$$

The first two moments of X are:

(3)
$$E(X)=\frac{6\left(\xi^2+1\right)+6\xi+6}{3\left(\xi+1\right)^2\psi},$$

(4)
$$E(X^2)=\frac{6\left(\xi+1\right)^2+2\left(6\xi+5\right)+4}{\left(\xi+1\right)^2\psi^2},$$

Therefore, the variance of the GQLD distribution is given by:

(5)
$$V(X)=E(X^2)-(E(X))^2=\frac{2\left(\xi^2+4\xi+2\right)}{\left(\xi+1\right)^2\psi^2}.$$

The corresponding reliability and hazard functions of the GQLD distribution are given, respectively by:

(6)
$$R_{GQLD}(x;\psi,\xi)=\frac{\left(\psi^3x^3+3\left(2\xi+1\right)\psi^2x^2+6\left(\xi+1\right)^2\psi x+6\left(\xi+1\right)^2\right)\mathrm{e}^{-\psi x}}{6\left(\xi+1\right)^2};x>0,\ \xi>-1,\ \psi>0,$$

(7)
$$H_{GQLD}(x;\psi,\xi)=\frac{6\psi^2\left(\frac{\psi^2x^3}{6}+\xi\psi x^2+\xi^2x\right)}{\psi^3x^3+3\left(2\xi+1\right)\psi^2x^2+6\left(\xi+1\right)^2\psi x+6\left(\xi+1\right)^2}.$$

The reversed hazard rate and odds functions for the GQLD distribution, respectively, are defined as

(8)
$$RH_{GQLD}(x;\psi,\xi)=\frac{\psi^2x\left(\psi^2x^2+6\xi\psi x+6\xi^2\right)}{6\left(\xi+1\right)^2\mathrm{e}^{\psi x}-\psi x\left(\psi x\left(\psi x+6\xi+3\right)+6\left(\xi+1\right)^2\right)-6\left(\xi+1\right)^2}.$$

and

(9)
$$O_{GQLD}(x;\psi,\xi)=\frac{6\left(\xi+1\right)^2\mathrm{e}^{\psi x}}{\psi^3x^3+3\left(2\xi+1\right)\psi^2x^2+6\left(\xi+1\right)^2\psi x+6\left(\xi+1\right)^2}-1.$$

3. Test statistics

In this section, we will discuss the suggested goodness of fit tests based on SRS and RSS methods.

3.1. **Using SRS.** Let $X_1, X_2, \ldots, X_m$ be a random sample from GQLD and let $\hat{\psi}_{SRS}$ and $\hat{\xi}_{SRS}$ be the maximum likelihood estimators of ψ and ξ, respectively, and let $f_0(.;\psi,\xi)$ be the probability distribution function of GQLD and $F_0(.;\psi,\xi)$ be the cumulative distribution function of GQLD. we consider the following test statistics:

- The Kullback-Leibler distance (Kullback and Leibler, 1951) between $g(x)$ and $g_0(x;\psi,\xi)$ is defined as

$$\begin{aligned}KL(g,g_0) &= \int_{-\infty}^{\infty} g(x)\log\left[\frac{g(x)}{g_0(x;\psi,\xi)}\right]dx,\\ &= -H(g)-\int_{-\infty}^{\infty} g(x)\log\left[g_0(x;\psi,\xi)\right].\end{aligned}$$

  Where $H(g)$ is the entropy defined by Shanon (1948) as

$$H(g)=-\int_{-\infty}^{\infty} g(x)\log(g(x))dx,$$

  and estimated by Vasicek (1976) by:

$$HV_{tm}=\frac{1}{m}\sum_{i=1}^{m}\log\left[\frac{m}{2t}(x_{(i+t)}-x_{(i-t)})\right],$$

  where t is integer less then $m/2$ known as window size and $x_{(i)}=x_{(m)}$ if $i>m$ and $x_{(i)}=x_{(1)}$ if $i<1$. This estimator converges in probability to $H(g)$ as $m,t\to\infty$ and $\frac{t}{m}\to 0$. Hence, the Kullback-Leibler test is given by Song (2002) by:

$$KL_{tm}=-HV_{tm}-\frac{1}{m}\sum_{i=1}^{m}\log\left[g_0(x_i,\hat{\psi}_{SRS},\hat{\xi}_{SRS})\right]$$

  where the distribution of KL_{tm} is free of ψ and ξ.
- The Kolmogorov-Smirnov test statistics Kolmogorov (1933) and Smirnov (1933):

$$KS=\max\left\{\max_{1\le i\le m}\left[\frac{i}{m}-G_0\ (x_{(i)},\ \hat{\psi}_{SRS},\hat{\xi}_{SRS})\right],\max_{1\le i\le m}\left[G_0(x_{(i)},\ \hat{\psi}_{SRS},\hat{\xi}_{SRS})-\frac{i-1}{m}\right]\right\}$$

- The Anderson-Darling test statistics Anderson and Darling (1954):

$$A^2=-2\sum_{i=1}^{m}\left\{\left(i-\frac{1}{2}\right)\log\left[G_0(x_{(i)},\ \hat{\psi}_{SRS},\hat{\xi}_{SRS})\right]+\left(m-i+\frac{1}{2}\right)\log\left[1-G_0(x_{(i)},\ \hat{\psi}_{SRS},\hat{\xi}_{SRS})\right]\right\}-m.$$

- The Cramer-von Mises test statistics Cramér (1928) and von Mises(1928):

$$W^2=\sum_{i=1}^{m}\left[G_0(x_{(i)},\hat{\psi}_{SRS},\hat{\xi}_{SRS})-\frac{2i-1}{2m}\right]^2+\frac{1}{12m}$$

- The Zhang (2002) test statistics:

$$Z_K=\max_{1\le i\le m}\left\{\left(i-\frac{1}{2}\right)\log\left[\frac{i-\frac{1}{2}}{mG_0(x_{(i)},\hat{\psi}_{SRS},\hat{\xi}_{SRS})}\right]+\left(m-i+\frac{1}{2}\right)\log\left[\frac{m-i-\frac{1}{2}}{m\left[1-G_0(x_{(i)},\hat{\psi}_{SRS},\hat{\xi}_{SRS})\right]}\right]\right\}$$

$$Z_A=-\sum_{i=1}^{m}\left\{\frac{\log\left[G_0(x_{(i)},\hat{\psi}_{SRS},\hat{\xi}_{SRS})\right]}{m-i+\frac{1}{2}}+\frac{\log\left[1-G_0(x_{(i)},\hat{\psi}_{SRS},\hat{\xi}_{SRS})\right]}{i-\frac{1}{2}}\right\},$$

$$Z_C=\sum_{i=1}^{m}\left[\log\left(\frac{G_0(x_{(i)},\hat{\psi}_{SRS},\hat{\xi}_{SRS})^{-1}-1}{\frac{(m-\frac{1}{2})}{(i-\frac{3}{4})^{-1}}}\right)\right]^2$$

4. Using RSS

Let $\{X_{[i]j}, i=1,2,\ldots,h; j=1,2,\ldots,r\}$ be a ranked set sample of size $M=hr$ from the GQLD and $z_{(1)}\le z_{(2)}\le\ldots\le z_{(M)}$ its corresponding ordered values and $\hat{\psi}_{RSS}$ and $\hat{\xi}_{RSS}$ be the maximum likelihood estimators of ψ and ξ, respectively, using RSS methods. Thus, above goodness of tests for RSS are:

- The test based on Kullback-Leibler distance is defined as

$$KL_{tm}^{RSS}=-HV_{tm}-\frac{1}{M}\sum_{i=1}^{M}\log\left[g_0(z_i,\hat{\psi}_{RSS},\hat{\xi}_{RSS})\right]$$

- The Kolmogorov-Smirnov test statistics:

$$KS=\max\left\{\max_{1\le i\le M}\left[\frac{i}{M}-G_0\ (z_{(i)},\ \hat{\psi}_{RSS},\hat{\xi}_{RSS})\right],\max_{1\le i\le M}\left[G_0(z_{(i)},\ \hat{\psi}_{RSS},\hat{\xi}_{RSS})-\frac{i-1}{M}\right]\right\}$$

- The Anderson-Darling test statistics:

$$A^2=-2\sum_{i=1}^{M}\left\{\left(i-\frac{1}{2}\right)\log\left[G_0(z_{(i)},\ \hat{\psi}_{RSS},\hat{\xi}_{RSS})\right]+\left(M-i+\frac{1}{2}\right)\log\left[1-G_0(z_{(i)},\ \hat{\psi}_{RSS},\hat{\xi}_{RSS})\right]\right\}-M.$$

- The Cramer-von Mises test statistics:

$$W^2=\sum_{i=1}^{M}\left[G_0(z_{(i)},\hat{\psi}_{RSS},\hat{\xi}_{RSS})-\frac{2i-1}{2M}\right]^2+\frac{1}{12M}$$

- The Zhang (2002) test statistics:

$$Z_K = \max_{1\le i\le M}\left\{\left(i-\frac{1}{2}\right)\log\left[\frac{i-\frac{1}{2}}{MG_0(z_{(i)},\hat{\psi}_{RSS},\hat{\xi}_{RSS})}\right]+\left(M-i+\frac{1}{2}\right)\log\left[\frac{M-i-\frac{1}{2}}{M\left[1-G_0(z_{(i)},\hat{\psi}_{RSS},\hat{\xi}_{RSS})\right]}\right]\right\}$$

$$Z_A = -\sum_{i=1}^{M}\left\{\frac{\log\left[G_0(z_{(i)},\hat{\psi}_{RSS},\hat{\xi}_{RSS})\right]}{M-i+\frac{1}{2}}+\frac{\log\left[1-G_0(z_{(i)},\hat{\psi}_{RSS},\hat{\xi}_{RSS})\right]}{i-\frac{1}{2}}\right\},$$

$$Z_C = \sum_{i=1}^{M}\left[\log\left(\frac{G_0(z_{(i)},\hat{\psi}_{RSS},\hat{\xi}_{RSS})^{-1}-1}{\frac{(M-\frac{1}{2})}{(i-\frac{3}{4})^{-1}}}\right)\right]^2$$

where $HV_{tm}^{RSS} = \dfrac{1}{M}\displaystyle\sum_{i=1}^{M}\log\left[\frac{M}{2t}(z_{(i+t)}-z_{(i-t)})\right]$, $G_0(z_{(i)},\psi,\xi)$ is the distribution function of GQLD distribution.

5. Simulation study

In this section, we investigate the performance of the power of the proposed goodness of fit test by using a Monte Carlo study. the study is based on 100,000 samples generated from the GQLD with scale parameter 1 and shape parameter 1 with different sample size using SRS and RSS design. the powers of tests based on KL_{tm} and KL_{tm}^{RSS} depend on the window size t. The problem of choosing the optimal values of t which maximizes the powers subject to m is still open in the field of entropy estimation. Therefore, in our simulations, we have used Grzegorzewski and Wieczorkowski (1999)'s heuristics formula for choosing t as: $t = [\sqrt{m}+0.5]$.

5.1. **Critical values.** Tables 1 present the critical values of the tests for the GQLD in RSS design for $\gamma = 0.01$. We take m from 2 to 9 and the set size h from 2 to 5.

5.2. **Power comparison.** In order to compare the powers of goodness of fit tests in SRS and RSS designs, we have considered twelve following distributions as alternative distributions:

- the Lindley distribution with parameter 1.
- the Lindley distribution with parameter 3.
- the quasi Lindley distribution with scale parameter 1 and shape parameter 1.
- the quasi Lindley distribution with scale parameter 1 and shape parameter 2.
- the quasi Lindley distribution with scale parameter 3 and shape parameter 1.
- the Exponential distribution with mean 1.
- the log-logistic distribution with scale parameter 2 and shape parameter 1.
- the Weibull distribution with scale parameter 1 and shape parameter 2.

- the Weibull distribution with scale parameter 2 and shape parameter 5.
- the power Lindley distribution with scale parameter 3 and shape parameter 3.
- the Uniform distribution on (0,1).
- the generalized Rayleigh distribution with scale parameter 1 and shape parameter 2.

Since the above tests statistic are location and scale invariant, the powers of these tests do not depend on the unknown parameters of GQLD. Tables 2 present the estimated powers, respectively, for $M = 10$ for $\gamma = 0.05$ using SRS and RSS methods. In RSS scheme, the value of h (set size) is taken to be 2 and 5, so we can observe the effect of increasing sample size while set size is fixed, and the effect of increasing set size while sample is fixed.
Remarks: Based on simulation results it can be observed that:

- The suggested RSS goodness-of-fit tests are more powerful than their SRS counterparts for all cases considered in this study. As an example considered the case when $M = 10$, $h = 2$ the quasi Lindley distribution with parameters(1,3) as an alternative the power values of the tests KS, A^2, W^2 based on RSS are 0.112, 0.382, 0.132 compared to 0.090, 0.270, 0.099 using RSS, respectively.
- The power of the goodness of fit tests increase as the set size k increase. As illustration, when N=20 for the Exponential distribution observe that $Z_K = 0.799$, $Z_A = 0.827$ and $Z_C = 0.878$ for $h = 2$ and for $h = 5$ $Z_K = 0.969$, $Z_A = 0.978$ and $Z_C = 0.970$.
- The power of the goodness of fit tests increase in the sample size. As an example, based on RSS with $h = 5$ for the Lindley distribution (1) , the power values of the Cramer-von Mises test are 0.343, 0.810, 0.998, respectively with $M = 10, 20, 40$.
- In most cases, the large power values are when the alternatives are power Lindley distribution (3,3) and generalized Rayleigh distribution (1,1).
- the power values of the suggested goodness-of-fit tests depend on the distribution parameters for the same test and sample size. As an example when $m = 20$ and $h = 2$ the power of the Anderson-Darling test are 0.777, 0.789, 0.714 for quasi Lindley distribution with parameters (1,1), (1,2), (3,1), respectively.

6. Real data example

In this section, we show the useful of our proposed RSS-goodness of fit by a real data example which present the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960), previously studied by Afify et al (2016). We present in Table 3 Akaike information criterion (AIC) introduced by Akaike (1974), Baysian information criterion (BIC) proposed by Schwarz (1978), HannanQuinn Information Criterion (HQIC) suggested by Hannan and Quinn (1979), Consistent Akaike Information Criterion (CAIC) by Bozdogan (1987), KS distance and its coreesponding p-value.

m	k	KL	KS	A^2	W^2	Z_K	Z_A	Z_C
2	2	2.290	0.573	18.995	0.319	2.017	4.528	13.900
2	3	1.676	0.477	40.109	0.311	2.436	4.278	18.141
2	4	1.339	0.410	69.032	0.297	2.664	4.101	19.710
2	5	1.138	0.359	105.460	0.280	2.770	3.958	20.963
3	2	1.363	0.499	41.284	0.358	2.618	4.379	17.251
3	3	0.982	0.406	88.292	0.340	2.977	4.108	20.630
3	4	0.783	0.347	152.853	0.321	3.093	3.936	22.114
3	5	0.660	0.300	234.643	0.292	3.162	3.810	22.431
4	2	1.099	0.446	71.878	0.382	3.000	4.245	19.698
4	3	0.801	0.361	155.261	0.366	3.369	3.999	22.520
4	4	0.644	0.302	269.224	0.330	3.402	3.830	23.599
4	5	0.546	0.260	412.653	0.288	3.379	3.719	24.108
5	2	0.937	0.408	110.830	0.399	3.277	4.133	21.073
5	3	0.686	0.328	240.371	0.377	3.567	3.906	24.122
5	4	0.555	0.272	417.146	0.334	3.598	3.756	24.814
5	5	0.479	0.234	640.590	0.288	3.548	3.660	25.030
6	2	0.818	0.380	158.162	0.421	3.544	4.061	22.533
6	3	0.609	0.302	343.864	0.390	3.742	3.838	24.843
6	4	0.495	0.248	595.812	0.327	3.744	3.700	25.503
6	5	0.431	0.214	918.307	0.286	3.695	3.614	25.881
7	2	0.695	0.358	213.650	0.435	3.755	3.994	23.693
7	3	0.502	0.281	464.991	0.395	3.852	3.778	25.319
7	4	0.404	0.230	806.901	0.326	3.847	3.654	26.148
7	5	0.342	0.199	1246.402	0.285	3.819	3.580	26.380
8	2	0.626	0.335	277.230	0.446	3.845	3.932	24.501
8	3	0.456	0.265	603.429	0.395	4.026	3.734	26.141
8	4	0.371	0.216	1049.537	0.325	3.967	3.623	26.534
8	5	0.316	0.186	1624.186	0.285	3.885	3.552	26.586
9	2	0.575	0.321	349.609	0.465	4.035	3.890	25.458
9	3	0.421	0.249	759.824	0.392	4.091	3.693	26.713
9	4	0.341	0.204	1325.100	0.325	4.019	3.595	27.388
9	5	0.296	0.175	2051.360	0.281	4.025	3.532	27.425

Table 1. Critical values of different tests of GQLD distribution for different values of (m, h) in RSS design, at significance level $\gamma = 0.01$.

It is clear from table 3 that the GQLD present a good fit to the data set. Hence we apply our proposed goodness of fit on a ranked set sample from the data of size $M = 10$ and set size $h = 5$. The sampled values are:

$$0.72, 1.63, 2.22, 1.71, 5.55, 0.10, 0.77, 1.53, 1.72, 4.32.$$

The estimated parameters using maximum likelihood method are $\hat{\psi} = 1.259$ and $\hat{\xi} = 0.671$, the values of all the test statistics are computed and given in Table 4. By comparing these values with the corresponding critical values, we observe that the null hypothesis that the data follow a WGQLD is not rejected by KL and Z_C at significance level of $\gamma = 0.05$.

Sampling scheme	Alternative distribution	Test Statistics						
		KL	KS	A^2	W^2	Z_K	Z_A	Z_C
SRS	Lindley (1)	0.124	0.091	0.286	0.101	0.267	0.291	0.362
	Lindley (3)	0.161	0.111	0.336	0.123	0.327	0.360	0.431
	Quasi Lindley (1,1)	0.125	0.092	0.287	0.100	0.270	0.293	0.363
	Quasi Lindley (1,2)	0.154	0.085	0.323	0.093	0.310	0.349	0.428
	Quasi Lindley (3,1)	0.117	0.090	0.270	0.099	0.258	0.282	0.347
	Exponential (1)	0.186	0.084	0.356	0.092	0.349	0.402	0.491
	Log-Logistic (2,1)	0.134	0.150	0.233	0.179	0.267	0.293	0.399
	Weibul (1,2)	0.183	0.074	0.337	0.080	0.335	0.391	0.479
	Weibul (2,5)	0.090	0.050	0.038	0.029	0.074	0.085	0.095
	Power Lindley (3,3)	0.814	0.907	0.982	0.980	0.873	0.921	0.915
	Uniform (0,1)	0.340	0.407	0.615	0.482	0.456	0.329	0.346
	Generalized Rayleigh (1,2)	0.759	0.880	0.970	0.969	0.846	0.930	0.911
RSS (k =2)	Lindley (1)	0.164	0.124	0.422	0.148	0.378	0.387	0.452
	Lindley (3)	0.190	0.113	0.411	0.133	0.391	0.420	0.480
	Quasi Lindley (1,1)	0.163	0.125	0.420	0.148	0.378	0.387	0.451
	Quasi Lindley (1,2)	0.207	0.111	0.457	0.136	0.426	0.450	0.520
	Quasi Lindley (3,1)	0.153	0.112	0.382	0.132	0.346	0.359	0.420
	Exponential (1)	0.253	0.105	0.494	0.132	0.478	0.516	0.590
	Log-Logistic (2,1)	0.181	0.161	0.310	0.198	0.384	0.404	0.489
	Weibul (1,2)	0.247	0.097	0.470	0.120	0.460	0.501	0.575
	Weibul (2,5)	0.117	0.135	0.124	0.124	0.147	0.195	0.165
	Power Lindley (3,3)	0.802	0.934	0.994	0.994	0.865	0.910	0.896
	Uniform (0,1)	0.327	0.481	0.725	0.590	0.481	0.312	0.306
	Generalized Rayleigh (1,2)	0.742	0.910	0.989	0.988	0.838	0.923	0.895
RSS (k =5)	Lindley (1)	0.266	0.266	0.840	0.343	0.631	0.631	0.600
	Lindley (3)	0.271	0.167	0.625	0.222	0.553	0.572	0.569
	Quasi Lindley (1,1)	0.267	0.266	0.838	0.346	0.633	0.628	0.600
	Quasi Lindley (1,2)	0.340	0.241	0.856	0.322	0.687	0.702	0.676
	Quasi Lindley (3,1)	0.240	0.232	0.742	0.300	0.572	0.577	0.555
	Exponential (1)	0.416	0.222	0.873	0.304	0.739	0.762	0.742
	Log-Logistic (2,1)	0.296	0.223	0.761	0.295	0.594	0.626	0.615
	Weibul (1,2)	0.411	0.213	0.854	0.294	0.728	0.751	0.732
	Weibul (2,5)	0.188	0.394	0.746	0.660	0.340	0.458	0.339
	Power Lindley (3,3)	0.864	0.991	1.000	1.000	0.942	0.954	0.933
	Uniform (0,1)	0.361	0.722	0.952	0.878	0.636	0.372	0.304
	Generalized Rayleigh (1,2)	0.799	0.983	1.000	1.000	0.923	0.961	0.933

Table 2. Power estimates of different goodness of fit tests in SRS and RSS designs for $M = 10$ and γ=0.05.

Table 3. AIC, AICc, BIC, HQIC,K-S distance and p-value for data set.

AIC	AICc	BIC	HQIC	K-S	p-value
209.5955	209.7694	214.1488	211.4082	0.092806	0.564624

KL	KS	A^2	W^2	Z_K	Z_A	Z_C
0.452	0.358	106.363	0.276	1.966	3.824	15.719

Table 4. Computing values of different test statistics in RSS.

7. Conclusion

In this paper, we developed goodness of fit test for GQLD using SRS and RSS methods. A simulation study was conducted to evaluate the power of the suggested goodness of fit tests. it is found that test based on RSS are more powerful than their based on SRS counterparts.

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ON ROBUST ESTIMATION FOR INCOMPLETE AND DEPENDENT DATA : SOME SIMULATIONS

GHELIEM ASMA AND GUESSOUM ZOHRA

Abstract. In this contribution we define the M-estimator of the regression function for associated and left-truncated data. Via a large simulations and through the influence function we illustrate that the M-estimator is more robust than the Nadaraya-Watson type estimator.

Keywords and phrases. M-estimator, truncated data, Associated data, outliers values , robust estimator , heavy-tailed error distribution, influence function.

1. Definition of the estimator

Let $(X_k, Y_k), 1 \leq k \leq N$ be a sequence of associated random vector , where X is a random vector of covariates, taking its values in $\mathbb{R}^d$ with (df) V and continuous density v and Y is a real random variable (rv) of interest with distribution function (df) F and T is the truncation variable with continuous df G, defined on the same probability space $(\Omega, F, \mathbb{P})$. We assume that T and (X, Y) are independent.
Under random left-truncation model (RLTM), the lifetime Y and T are observable only when $Y \geq T$, and $n \leq N$. Let $\mu =: \mathbf{P}(Y \geq T)$ be the probability to observe Y.

Under RLTM, we denote by m(x)(robust regression) the implicit solution with respect to (w.r.t) s of

$$H(x,s) := \frac{1}{\mu}\mathbb{E}[\psi(Y_1 - s)|X_1 = x)]v(x) = 0$$

$\psi(.)$ is a bounded function.

The M-estimator of $m(x)$, denoted by $\hat{m}_n(x)$, is defined by the implicit solution w.r.t. s of

$$\hat{H}_n(x,s) := \frac{1}{nh_n^d}\sum_{i=1}^{n}\frac{1}{G_n(Y_i)}K_d\left(\frac{x - X_i}{h_n}\right)\psi(Y_i - s) = 0,$$

where: K_d is a kernel function on $\mathbb{R}^d$ and h_n is a sequence of positive real numbers which goes to zero as n goes to infinity and $G_n(x)$ is the well known product limit estimator of $G(x)$, proposed by Lynden-Bell (1971).

2. Simulation

A large simulation study is carried out to comfort the good behavior of the M-estimator. We show that the proposed estimator performs better than the Nadaraya-Watson estimator first, by looking at their proximity to the true regression function in dimension one and in dimension two. Thereafter and to highlight the robustness of our estimator, we consider a regression model with a heavy-tailed error distribution and we compare the global mean square error of the two estimators. Next, we investigate the behavior of the two estimators in the presence of outliers using the influence function. We show that the M-estimator is much more robust to the presence of outliers.

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ON THE ESTIMATION OF MARKOV-SWITCHING PERIODIC GARCH MODEL

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Abstract. In this work, we will focus on the estimation of the markov-switching periodic GARCH model, which is a GARCH process with time-varying parameters governed by a hidden markov chain and a periodicity structure. This model is more flexible and it allows capturing stylized facts like heavy tails, asymmetry and volatility clustering. We propose a maximum likelihood procedure based on the collapsing procedure proposed previously in the literature. We perform also a simulation study and an application to the Algerian exchange rate.

2010 Mathematics Subject Classification. 62M10, 62M05, 65C35

Keywords and phrases. Markov switching GARCH model, Periodicity, Path dependence, Collapsing procedure, Maximum likelihood, Exchange rates.

1. Define the problem

Modelling volatility has received much interest from researchers since the introduction of the ARCH model by Engle (1982) and its generalization by Bollerslev (1986). In fact, these models represent powerful tools in the study, forecast and analysis of financial and macroeconomic phenomenon since they capture many stylized facts observed in their data, like heavy-tailed marginal distributions, asymmetry and volatility clustering. However, other patterns as multimodality and regime changes remain uncaptured by these models, hence the necessity to extend them to other classes like the mixture models and the Markov-Switching models.

The Markov-Switching (in short MS) models proposed firstly by Hamilton (1989) have attracted many researchers and practitioners due to their flexibility. Indeed, Cai (1994) and Hamilton and Susmel (1994) proposed MS-ARCH. Gray (1996) proposed a simplified version of MS-GARCH. Thereby, numerous studies have been implemented, devoted to MS-GARCH (see e.g. Klaassen, 2002; Haas et al., 2004; francq and Zaköian, 2005).

On the other hand, many economic and financial time series display seasonal variation that should be taken into account. Many researchers emphasized the need to combine periodicity with the GARCH-type models discussed above (see e.g. Bollerslev and Ghysels, 1996; Franses and Paap, 2000). Such observations have led to the development of some mixture models that explicitly incorporate this periodicity in parameter structure. Bentarzi and Hamdi (2008) introduced a mixture periodic ARCH (MPARCH) and successfully applied it to model the S&P 500 stock price closing index. Hamdi and Souam (2013, 2018) have discussed two mixture periodic GARCH models that constitute very flexible and more parsimonious classes of periodic time series models of the conditional variance comparatively to MPARCH models. Recently, Aliat and Hamdi (2019) have proposed the class of Markov-Switching periodic GARCH models and provide an estimation method based on the generalized method of moments.

It should be pointed out that the estimation of MS-PGARCH model is a challenging task because the conditional variance depends on the entire history of regimes generated by the markov chain and thus all the past information which causes an exponential increase in the number of possible paths and thus an explosive intractable likelihood.

The main contribution of this work is to propose an estimation approach for the MS-PGARCH model based on the work of Augustyniak et al. (2018). We will carry out a simulation study and perform an empirical analysis on the Algerian exchange rate. We will also compare our method to the GMM proposed by Aliat and Hamdi (2019).

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ON THE LOCAL LINEAR MODELIZATION OF THE CONDITIONAL MODE FOR FUNCTIONAL AND ERGODIC DATA

SOMIA AYAD AND SAÂDIA RAHMANI

Abstract. In this paper, we estimate the conditional mode using the local linear approach. We treat the case when the regressor is valued in a semi-metric space, the response is a scalar and the data are observed as ergodic functional times series. Under this dependence structure, we state the almost complete consistency (a.co.) with rates of the constructed estimator. Moreover, an application on real data has been conducted in order to highlight the superiority of our method to the standard kernel method, in the functional framework.

2010 Mathematics Subject Classification.62G05, 62G08, 62G20.

Keywords and phrases. Ergodic data, functional data, local linear estimator.

1. The model and its estimate

Let $Z_i = (X_i, Y_i)_{i=1,\dots,n}$ be an $\mathcal{E} \times \mathbb{R}$-valued measurable strictly stationary process, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where $\mathcal{E}$ is a semi-metric space, and d denotes the semi-metric. Furthermore, we assume that there exists a regular version of the conditional distribution of Y given X, which is absolutely continuous with respect to the *Lebesgue* measure on $\mathbb{R}$, and has a twice continuously differentiable probability density function denoted by $f^X(Y)$, has bounded density. Moreover, we suppose that the conditional density $f^X(Y)$ is unimodal in some fixed compact $\mathscr{C}$ and the conditional mode, denoted by $\Theta(x)$ is defined by:

$$\Theta(x) = \arg\sup_{y\in\mathscr{C}} f^x(y).$$

Now, we assume that the underlying process Z_i is functional stationary ergodic, a natural and usual local linear estimator of $\Theta(x)$ is given by:

$$\widehat{\Theta}(x) = \arg\sup_{y\in\mathscr{C}} \widehat{f}^x(y). \tag{1}$$

Where $\widehat{f}^x(.)$ is the local linear estimator of $f^X(.)$ defined by:

$$\widehat{f}^x(y) = \frac{\sum_{j=1}^{n} \Gamma_j K_j J_j}{h_J \sum_{j=1}^{n} \Gamma_j K_j}, \tag{2}$$

with

$$\Gamma_j = \sum_{i=1}^{n} \rho_i^2 K_i - \left(\sum_{i=1}^{n} \rho_i K_i \right) \rho_j,$$

$$\rho_i = \rho(X_i, x), K_i = K\left(\frac{\delta(x, X_i)}{h_K} \right) \text{and } J_j = J\left(\frac{y - Y_j}{h_J} \right),$$

where K and J are kernels functions and $h_K = h_{K,n}$ (resp. $h_J = h_{J,n}$) is a sequence of positive real numbers. $\rho(.,.)$ and $\delta(.,.)$ are known bi-functional operators defined from $\mathcal{E}^2$ into $\mathbb{R}$ such that $|\delta(x,z)| = d(x,z)$ and $\rho(z,z) = 0, \forall z \in \mathcal{E}$. (see Barrientos et *al.* [2] for some examples of these two locating functions). Such fast version of functional local linear estimation has been proposed by Demongeot et *al.* [8] under the strong mixing condition usually assumed in functional time series analysis.

We recall that the estimator (2) is obtained from the following minimization procedure:

$$\min_{(a_0,a_1)\in\mathbb{R}^2} \sum_{i=1}^{n} \left(\frac{1}{h_J} J\left(\frac{y - Y_i}{h_J} \right) - a_0 - a_1 \rho(X_i, x) \right)^2 K\left(\frac{\delta(x, X_i)}{h_K} \right). \tag{3}$$

The main purpose of this paper is to study the nonparametric estimate $\widehat{\Theta}(x)$ of the conditional mode $\Theta(x)$ by the local linear approach. Recall that these questions in infinite dimension are particularly interesting, not only for the fundamental problems they formulate, but also for many applications, (see for instance Dabo and Laksaci [6] and Dabo et *al.* [5]).

In the statistical literature, several papers have been devoted to the study of some properties of the nonparametric stationary ergodic processes estimators (see for instance, Didi and Louani [10] in the case of complete data and Chaouch et *al.* [4] for right censored ones).

2. Main results

In the following, for any fixed x in $\mathcal{F}$, $\mathcal{N}_x$ denotes a fixed neighborhood of x; and let us denote by $\phi_x(r_1, r_2) = \mathbb{P}(r_2 \leq \delta(X, x) \leq r_1)$ the small ball probability function.

The following proposition establishes the almost complete convergence (with rate) of the conditional density estimator $\widehat{f^x}(y)$. This proposition which is of interest by itself used, as an intermediate result, to prove our main result given in Theorem 2.2.

Proposition 2.1. *Under some structural regularity and technical assumptions, we have*

$$\sup_{y\in\mathscr{C}} |\widehat{f^x}(y) - f^x(y)| = O\left(h_K^{b_1}\right) + O\left(h_J^{b_2}\right) + O\left(\sqrt{\frac{\varphi_x(h_K)\log n}{n^2 h_J \phi_x^2(h_K)}} \right), \quad a.co.$$

Where b_1, b_2 are positive constants linked to the Lipchitz condition and $\varphi_x(h_K) = \sum_{i=1}^{n} \phi_{i,x}(h_K)$.

Theorem 2.2. *Under assumptions of Proposition 2.1, we have*

$$|\widehat{\Theta}(x) - \Theta(x)| = O\left(h_K^{\frac{b_1}{j}}\right) + O\left(h_J^{\frac{b_2}{j}}\right) + O\left(\left(\frac{\varphi_x(h_K)\log n}{n^2 h_J \phi_x^2(h_K)}\right)^{\frac{1}{2j}}\right), \quad a.co.$$

Where j is the order of derivative of conditional density f^x.

3. A real data application

In this part, we apply our thereticol results to the problem of ozone concentration forecasting by the prediction the total ozone in one day ahead using the conditional mode estimation. Precisely, we consider the the ozone data collected in Marylebone road monitoring site. In this application study we focus on the hourly measurements of this polluting gas during the 2018-year. The data of this example is provided by the website https://www.airqualityengland.co.uk/.

In this context, we apply the local linear mode estimation to predict the total ozone concentration in one day ahead the whole daily curves (one day before). Indeed, for the functional random variables $(X_i)_{i=1,\dots,N}$ defined by: $\forall t \in [0, b[,\ X_i(t) = Z_{((i-1)b+t)/N}$, Z_t designs the ozone concentration for 8736 hours between 01/01/2018 and 31/12/2018. We cut this functional time series in $N + 1 = 364$ pieces X_i of 24 hours (one day). These functionals variables X_i are presented by the following figure (Fig. 1.)

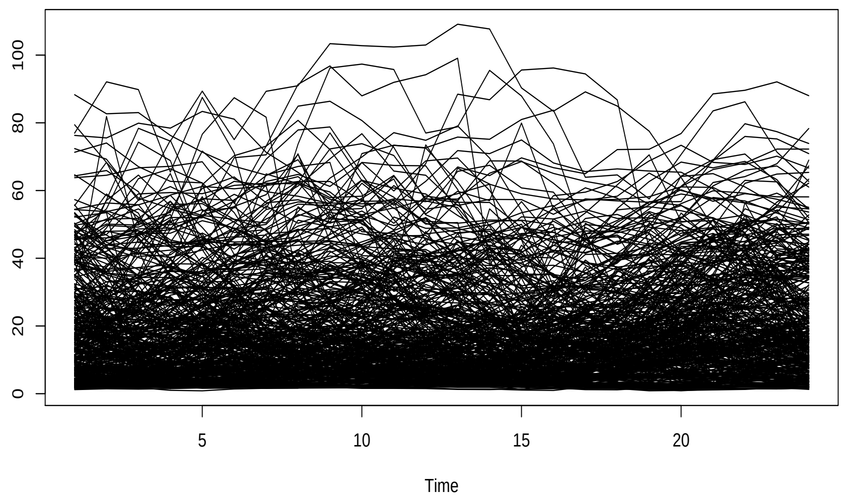

Fig. 1. Hourly ozone concentration of the year 2018.

The scalar response variable Y is defined by $Y_i = \sum_{h=0}^{23} X_{i+1}(h)$. For this comparison study we compute both estimators (Ker and LL) in its optimal conditions. In particular, we choose the optimal bandwidths (h_K, h_J) locally by the cross-validation method on the k-nearest neighbors with respect the following MSE-criterion $MSE(Ker) = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \widetilde{\Theta}^{-i}(X_i))^2$, and $MSE(LL) = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \widehat{\Theta}^{-i}(X_i))^2$ where $\widetilde{\Theta}^{-i}$ (resp. $\widehat{\Theta}^{-i}$) designs the leave-one-out kernel (resp. local linear) estimator of the conditional mode. We use quadratic kernel $J(x) = K(x) = \frac{3}{4}(1-x^2)1\!\!1_{[0,1]}$. The semi-metric d_{PCA} based on the $m = 3$ first eigenfunctions of the empirical covariance operator associated to the $m = 3$ greatest eigenvalues is more adapted of these discontinuous curves and we take $\rho = \delta$ (for the LL estimator).

we compute the kernel estimator (Ker) defined by:

$$\widetilde{\Theta}(x) = \sup_{y \in \mathscr{C}} \widetilde{f}^x(y)$$

where

$$\widetilde{f}^x(y) = \frac{h_J^{-1}\sum_{i=1}^{n} K(h_K^{-1}d(x,X_i))J(h_J^{-1}(y-Y_i))}{\sum_{i=1}^{n} K(h_K^{-1}d(x,X_i))},$$

and our LL estimator $\widehat{\Theta}(x)$.

Now, in order to compare both methods we split our data into two subsets I_1 and I_2. The 244 observations $(X_j, Y_j)_{j\in I_1}$ will be our statistical sample from which are calculated the estimators and the 120 remaining observations $(X_i, Y_i)_{i\in I_2}$ are considered as the testing sample. Next, we use the following algorithm:

- *Step 1.* For each curve X_j in the input sample we approximate the associated response variable Y_j by

$$\widehat{Y}_j = \widetilde{\Theta}(X_j)$$

  and

$$\widehat{Y}_j = \widehat{\Theta}(X_j).$$

- *Step 2.* For each X_{new} in the testing sample, we put

$$i^* = \arg\min_{j\in I_1} d(X_{new}, X_j).$$

- *Step 3.* For each X_{new} we put

$$h_K = \text{ the optimal bandwidth parameter associated to } X_i^*$$

  and

$$h_J = \text{ the optimal bandwidth parameter associated to } Y_i^*$$

- *Step 4.* We predict Y_{new} by

$$\widehat{Y}_{new} = \widetilde{\Theta}(X_{new})$$

  and

$$\widehat{Y}_{new} = \widehat{\Theta}(X_{new}).$$

- *Step 5.* We calculate the prediction error s expressed by

$$\frac{1}{120}\sum_{i\in I_1}(Y_i - \widehat{T}(X_i))^2,$$

  where $\widehat{T}$ means either the kernel estimator or the local linear one.
- *Step 6.* We divide again our observations in the two subsets I_1 and I_2 and we repeat the step 1-5.
- *Step 7.* We repeat the Step 6 several times.
- *Step 8.* We end this analysis by plotting the box-plot of the mean square errors of each method.

The comparisons study is carried out by repeating the algorithm 60 times with random splitting of the observations between training and testing sample. We point out that the scatter-plots indicates that the local linear method is significantly better than the kernel method.

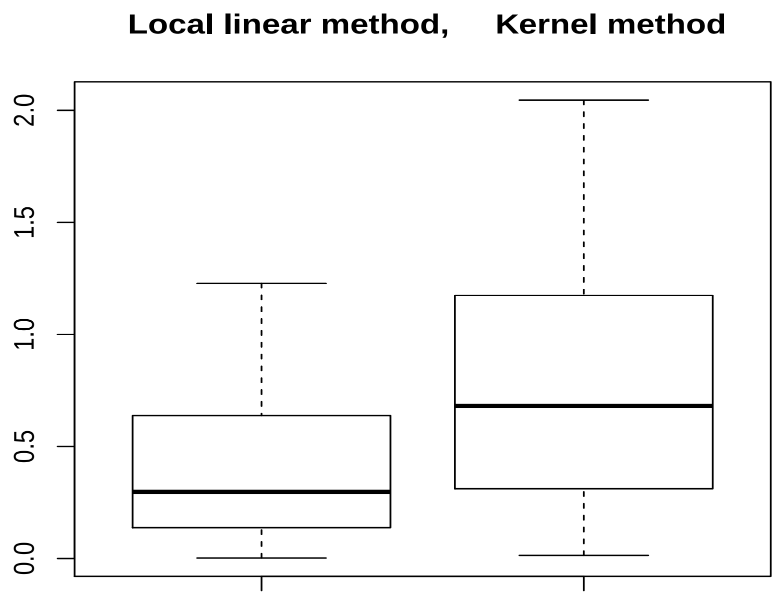

Fig. 2. Comparison of the Ozone concentration prediction between the kernel method and the local linear approach

4. Conclusion

It should be noticed that, both methods, Nadaraya-Watson and local linear fit, can be thought as two particular cases of the local polynomial smoother with order $k = 0$ and $k = 1$ respectively. Nevertheless, the first one suffers from a larger bias than the local linear estimator.
In this paper we confirm the superiority of the local linear approach over the kernel method through the establishment an asymptotic property of our estimator, in terms of the almost-complete convergence with rates. Moreover, the usefulness of our results is illustrated through its application to the ozone data.

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ON A MULTISERVER QUEUEING SYSTEM WITH CUSTOMERS' IMPATIENCE UNTIL THE END OF SERVICE UNDER SINGLE AND MULTIPLE VACATION POLICIES

MOKHTAR KADI, AMINA ANGELIKA BOUCHENTOUF,
AND LAHCENE YAHIAOUI

ABSTRACT. This paper deals with a multi server queueing system with Bernoulli feedback and impatient customers (balking and reneging) under synchronous multiple and single vacation policies. Reneged customers may be retained in the system. Using PGFs probability generating functions technique, we formally obtain the steady-state solution of the proposed queueing system. Further, important performance measures and cost model are derived. Finally, numerical examples are presented.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 2000 Mathematics Subject Classification. Primary 60K25; Secondary 68M20; Thirdly 90B22

KEYWORDS AND PHRASES. Queueing models; synchronous vacation; impatient customers; Bernoulli feedback.

1. DEFINE THE PROBLEM

Consider a $M/M/c$ queueing model with Bernoulli feedback, balking, reneging and retention of reneged customers. Customers arrive into the system according to a Poisson process with arrival rate λ.

the service time, the vacation time, and the impatience time (in vacation and busy period) are assumed to be exponentially distributed with rates μ, ϕ and (ξ_0, ξ_1) respectively. The service discipline is FCFS and there is a infinite space for customers to wait. The servers take vacation synchronously once the system becomes empty, and they also return to the system as one at the same time.

In this paper we consider two vacation type queueing models

Model I: Single station vacation policy.

Model II: Multiple station vacation policy.

We suppose that the customers timers are independent and identically distributed random variables and independent of the number of waiting customers.

Each reneged customer may leave the system without getting service with probability α and may remain in the queue for his service with probability $\bar{\sigma} = (1-\sigma)$. A customer who on arrival finds at least one customer (resp. c customers) in the system, when the servers are on vacation period (resp. busy period) either decides to enter the queue with probability θ or balk

with probability $\overline{\theta} = 1 - \theta$.
After completion of each service, the customer can either leave the system definitively with probability β or come back to the system and join the end of the queue with probability β', where $\beta + \beta' = 1$. Let $L(t)$ be the number of customers in the system at time t, and $J(t)$ represents the status of the server at time t, such that

$$J(t) = \begin{cases} 0, & \text{all the servers are taking a vacation at time } t; \\ 1, & \text{the servers are busy at time } t. \end{cases}$$

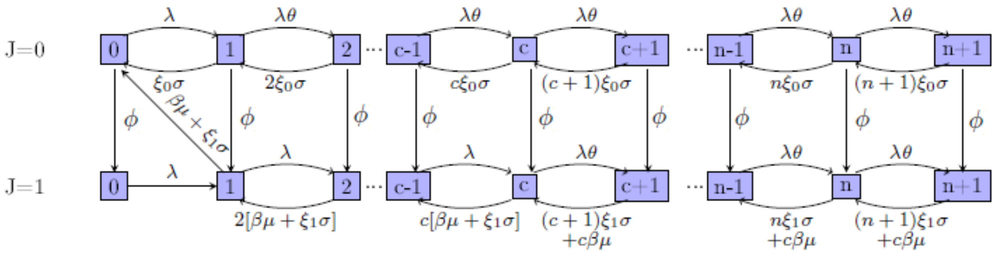

FIGURE 1. State-transition diagram for Model I

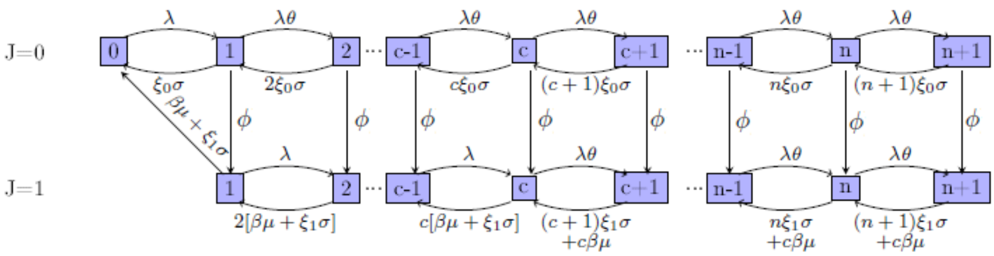

FIGURE 2. State-transition diagram for Model II

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On the behavior of Lynden Bell estimator under association

This study is devoted to assess the asymptotic behavior of the Lynden-Bell estimator for the truncating low under left truncated when the process satisfying the association dependence in the sense of Esary et al. (1967). This work is a complementary result to the work of Guessoum et al. (2012). These authors established a strong uniform consistency rate of the Lynden-Bell estimator of the marginal distribution function of the interest variable. The accuracy of the studied estimates is checked by a simulation study.

ON THE ESTIMATION OF THE MEAN OF A MULTIVARIATE NORMAL DISTRIBUTION UNDER THE BALANCED LOSS FUNCTION

ABDELKADER BENKHALED

Abstract. In this work, we deal with the shrinkage estimators of the mean θ of a multivariate normal distribution $X \sim N_p\left(\theta, \sigma^2 I_p\right)$, where the parameter σ^2 is unknown and estimated by S^2 ($S^2 \sim \sigma^2\chi^2_n$). For compared between two estimators, we use the risk associated to the balanced loss function. Firstly, we establish the minimaxity of the considered estimators when the dimension of the parameters space is finite. Secondly, when the dimension of the parameters space p and the sample size n tend to infinity, we study the asymptotic behavior of risks ratio of these estimators to the maximum likelihood estimator (MLE). In the end, we conduct a simulation study that show the performance of the considered estimators.

2010 Mathematics Subject Classification. Primary: 62F12. Secondary: 62C20.

Keywords and phrases. Balanced loss function, James-Stein estimator, Minimaxity, Multivariate normal random variable, Shrinkage estimator , Risks ratio.

1. Define the problem

Let $X \sim N_p\left(\theta, \sigma^2 I_p\right)$, where the parameter σ^2 is unknown and estimated by S^2 ($S^2 \sim \sigma^2\chi^2_n$). Our aim is to estimate the unknown parameter θ by the shrinkage estimators method.
We consider the estimator

$$\delta_a = \left(1 - a\frac{S^2}{\|X\|^2}\right)X = X - a\frac{S^2}{\|X\|^2}X, \tag{1}$$

where a is a real parameter.

In the first part, we study the minimaxity of the estimator δ_a. For $a = \frac{(1-\omega)(p-2)}{n+2} := \alpha$ we obtain the James-Stein estimator

$$\delta_{JS} = \left(1 - \alpha\frac{S^2}{\|X\|^2}\right)X, \tag{2}$$

and we deduce the positive-part of James-Stein estimator

$$\delta^+_{JS} = \left(1 - \alpha\frac{S^2}{\|X\|^2}\right)^+ X = \left(1 - \alpha\frac{S^2}{\|X\|^2}\right) X\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\leq 1} \tag{3}$$

where $\left(1 - \alpha\frac{S^2}{\|X\|^2}\right)^+ = \max\left(0, 1 - \alpha\frac{S^2}{\|X\|^2}\right)$ and $\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\leq 1}$ denoted the indicating function of the set $(\alpha\frac{S^2}{\|X\|^2} \leq 1)$. Moreover we study the domination of δ^+_{JS} to δ_{JS}.

In the second part, we treat the asymptotic behavior of risks ratios of James-Stein estimator and the positive-part of the James-Stein estimator to the MLE, when the dimension p tends to infinity and the sample size n is fixed on one hand, and on the other hand when p and n tend simultaneously to infinity.

Finally, we graphically illustrate the obtained results.

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Title : On the existence and stability of solutions of stochastic differential systems driven by G-Brownian motion

Autors : El-Hacène Chalabi

Abstract :

In this paper, we study the Carathéodory approximate solution for the folowing stochastic differential systems driven by G-Brownian motion.

$$\begin{cases} X_1(t) = & X_1(0) + \int_0^t f_{1,1}(s, X_1(s), \dots, X_n(s))\, ds + \\ & \int_0^t f_{2,1}(s, X_1(s), \dots, X_n(s))\, d\langle B\rangle(s) + \\ & \int_0^t f_{3,1}(s, X_1(s), \dots, X_n(s))\, dB(s) \\ \vdots & \vdots \\ X_n(t) = & X_n(0) + \int_0^t f_{1,n}(s, X_1(s), \dots, X_n(s))\, ds + \\ & \int_0^t f_{2,n}(s, X_1(s), \dots, X_n(s))\, d\langle B\rangle(s) + \\ & \int_0^t f_{3,n}(s, X_1(s), \dots, X_n(s))\, dB(s) \end{cases}$$

Based on the Carathéodory approximation scheme, we prove under some suitable conditions that our system have a unique solution and show that the Carathéodory approximate solutions converges to the solution of the system. Moreover, we prove a stability theorem for our system.

Keywords : G-expectation, G-brownian motion, G-stochastic differential equations, Carathéodory approximation scheme.

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ON THE LOCAL TIME OF A REFLECTING BROWNIAN MOTION

A. BENCHÉRIF-MADANI AND N. KACEM

Abstract. Let $X(t)$ be a Brownian motion reflecting at zero. We prove that the time spent below zero (up to t) by the penalized Brownian motion, normalized by a square root, converges in probability to the local time of reflecting Brownian motion at zero. That is, consider the reflecting Brownian motion X_t in $\mathbb{R}^+$ starting at 0 for convenience

$$X_t = B_t + L_t,$$

in which B is a standard Brownian motion and L_t is the local time at 0. Set $\beta(x) = -x/\delta$ for $x \leq 0$ and $\beta = 0$ otherwise. Consider the penalized diffusions X_t^δ, also starting from 0, i.e. $X_t^\delta = B_t - \int_0^t \beta(X_s^\delta)ds$ and the time spent below zero (up to t)

$$\mathcal{T}^\delta(t) = \int_0^t I_{\left\{X_s^\delta \leq 0\right\}} ds.$$

We prove that there exists a constant $c_0 > 0$ such that for all t as $\delta \to 0$ we have $\dfrac{\mathcal{T}^\delta(t,0)}{\sqrt{\delta}} \to c_0 L(t,0)$ in probability.

Applications include Finance for example and PDEs etc. .

2010 Mathematics Subject Classification. 60xx, 35xx.

Keywords and phrases. Reflecting diffusion, Local time, PDEs.

1. Setting of the problem

Let X be a linear diffusion in $\mathbb{R}^+$ reflecting at zero. The actual knowledge of the local time at the end-point zero, say L_t, is important in many pratical problems such as Finance for example or PDEs with boundary conditions, especially of Neumann type. Recall that this is the probabilistic counterpart of the study of, .e.g., the PDE $\partial_t u(t,x) = a(x)\partial_{xx}u(t,x)+b(x)u(t,x)\partial_x u(t,x)$ in $x > 0$ together with $\partial_x u(t,x) = d$ at zero. Indeed, by using simulations and probabilistic representations for the solution of the above deterministic PDE, we often can derive an approximate solution exactly as is routine work with ordinary numerical methods for PDEs, see e.g. [**?**].

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On the solution of Mckean-Vlasov equations via small delays

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Abstract

We study the strong convergence of the Carathéodory numerical scheme for a class of nonlinear McKean-Vlasov stochastic differential equations (MVSDE). We prove, under Lipschitz assumptions, the convergence of the approximate solutions to the unique solution of the MVSDE. Moreover, we show that the result remains valid, under continuous coefficients, provided that pathwise uniqueness holds. The proof is based on weak convergence techniques and the Skorokhod embedding theorem. In particular, this general result allows us to construct the unique strong solution of a MVSDE by using the Carathéodory numerical scheme. Examples under which pathwise uniqueness holds are given.

Keywords: McKean-Vlasov equation; mean-field equation; carathéodory numerical scheme; wasserstein distance; delay equation; tightness; pathwise uniqueness; strong solution.

2010 Mathematics Subject Classification. 60H10, 60H07, 49N90

OPTIMUM COST ANALYSIS FOR A DISCRETE-TIME MULTISERVER WORKING VACATION QUEUEING SYSTEM WITH CUSTOMERS' IMPATIENCE

LAHCENE YAHIAOUI, AMINA ANGELIKA BOUCHENTOUF, AND MOKHTAR KADI

Abstract. In this work, we deal with a discrete-time finite-capacity multiserver queueing system with Bernoulli feedback, synchronous multiple and single working vacations, balking, and reneging during both busy and working vacation periods. A reneged customer can be retained in the system by employing certain persuasive mechanism for completion of service. Using recursive method, the explicit expressions for the stationary state probabilities are obtained. Based on the system performance measures, a cost model is formulated. Then, the optimization of the model is carried out using quadratic fit search method (QFSM).

Keywords and phrases. Multiserver queueing systems, synchronous vacation, impatient customers, Bernoulli feedback, cost model, optimization.

1. Introduction

Discrete-time queueing systems attainted a significant importance because of their wide applicability in the performance analysis of telecommunication systems. They are very appropriate for modeling and analyzing digital communication systems. Typical examples are synchronous communication systems (slotted ALOHA), packet switching systems with time slots and broad integrated services digital networks (B-ISDN) based on asynchronous transfer mode (ATM) technology, as the information contained in the B-ISDN is routed through discrete units. More details on discrete-time queues are given in the survey paper of [3] and the monographs of [2, 5, 1]. In this work, we consider a finite-buffer discrete-time multiserver queueing system with Bernoulli feedback, single and multiple working vacation policies, balking, reneging during both normal busy and working vacation periods, and retention of reneged customers under late arrival system with delayed access (LASDA).

2. The model

We suppose that inter-arrival times A of customers, service times during both normal and working vacation periods, vacation times, impatience time are independent and geometrically distributed with rates $\lambda, \mu, \nu, \theta$, and ξ respectively. The system is composed of c servers. The discipline of system is FCFS . The capacity of the system is taken as finite (say, N). The arriving customer may join the queue with probability ϑ_n or balk with a complementary probability $\overline{\vartheta}_n = 1 - \vartheta_n$, with $0 \leq n \leq N$. In addition, we suppose that $0 \leq \vartheta_{n+1} \leq \vartheta_n \leq 1,\ c \leq n \leq N-1,\ \vartheta_0 = 1, ..., \vartheta_{c-1} = 1,$ and $\vartheta_N = 0$. A synchronous vacation is considered, that is, the servers go all together on working vacation, under multiple working vacation (MWV) or single working vacation policy. Further, customers may get impatient and leave the queue without getting service with some probability σ. The reneged customer can be retained in the queueing system with probability $\overline{\sigma} = 1 - \sigma$. After completion of each service, a customer can either join the end of the queue for another regular service with probability $\overline{\beta}$ or leave the system with

1

probability β, where $\overline{\beta} = 1 - \beta$. Further, note that at one slot we may have, one arrival, a departure from a service, and a departure from the queue as reneged customer.

3. Steady-state analysis

Let δ be the indicator function:

$$\delta = \begin{cases} 1, & \text{for the single working vacation model,} \\ 0, & \text{for the multiple working vacation model.} \end{cases}$$

At steady-state, $\pi_{i,0},\ 0 \leq i \leq N$ denotes the probability that there are i customers in the system when the servers are in working vacation period and $\pi_{i,1},\ 1-\delta \leq i \leq N$ is the probability that there are i customers in the system when the servers are in normal busy period.
Based on the one-step transition analysis, the steady-state equations can be given as

$$\begin{aligned}
\pi_{0,0} &= M_0(\eta)\pi_{0,0} + A_1(\eta)\pi_{1,0} + C_2 d_2(\eta)\pi_{2,0} + A_1(\mu)\pi_{1,1} + C_2(\mu)\pi_{2,1}, \\
\pi_{i,0} &= \overline{\theta}\left[A_{i+1}(\eta)\pi_{i+1,0} + B_{i-1}(\eta)\pi_{i-1,0} + C_{i+2}(\eta)\pi_{i+2,0} + M_i(\eta)\pi_{i,0}\right], \ i = 1...N, \\
\pi_{0,1} &= \overline{\lambda}\pi_{0,1} + \theta\overline{\lambda}\delta\pi_{0,0}, \\
\pi_{1,1} &= M_1(\mu)\pi_{1,1} + A_2(\mu)\pi_{2,1} + C_3(\mu)\pi_{3,1} + \lambda\delta\pi_{0,1} + \theta\left[M_1(\eta)\pi_{1,0} + \lambda\pi_{0,0} + A_2(\eta)\pi_{2,0}\right. \\
&\quad \left.+ C_3(\eta)\pi_{3,0}\right], \\
\pi_{i,1} &= M_i(\mu)\pi_{i,1} + B_{i-1}(\mu)\pi_{i-1,1} + A_{i+1}(\mu)\pi_{i+1,1} + C_{i+2}(\mu)\pi_{i+2,1} + \theta\left(M_i(\eta)\pi_{i,0} + B_{i-1}(\eta)\right. \\
&\quad \left.\times\pi_{i-1,0} + A_{i+1}(\eta)\pi_{i+1,0} + C_{i+2}(\eta)\pi_{i+2,0}\right), \ i = 1...N,
\end{aligned}$$

where $A_i(x) = \overline{\lambda\vartheta_i}(d_i(x)\,\overline{r}_i + \overline{d}_i(x) r_i) + \lambda\vartheta_i d_i(x) r_i;\ 1 \leq i \leq N$

$B_i(x) = \lambda\vartheta_i\,\overline{d}_i(x)\,\overline{r}_i;\ \ 0 \leq i \leq N-1,\ \ C_i(x) = \overline{\lambda\vartheta_i} d_i(x) r_i;\ \ 2 \leq i \leq N.$

$$M_i(x) = \begin{cases} \overline{\lambda}(1-\theta\delta) & ;\ i = 0 \\ \overline{\lambda\vartheta_i}\,\overline{d}_i(x)\,\overline{r}_i + \lambda\vartheta_i(d_i(x)\,\overline{r_i} + \overline{d}_i(x) r_i) & ;\ 1 \leq i \leq N \end{cases}$$

where

$$d_n(x) = \begin{cases} 1 - \overline{x\beta}^n, & \text{if } 1 \leq n \leq c-1, \\ 1 - \overline{x\beta}^c, & \text{if } c \leq n \leq N, \end{cases} \quad \text{and } r_n = \begin{cases} 0, & \text{if } 1 \leq n \leq c, \\ 1 - \overline{\xi\sigma}^{n-c}, & \text{if } c+1 \leq n \leq N. \end{cases}$$

We obtain the steady-state probabilities $\pi_{i,0},\ 0 \leq i \leq N$ and $\pi_{i,1},\ 1-\delta \leq i \leq N$, using a recursive method, the results of the steady-state probabilities and performance measures find in [4].

4. Optimisation analysis

The total expected cost per unit time of the system, Γ, is given as

$$\Gamma = C_b P_b + C_{wv} P_{wv} + C_{id} P_{id} C_{Rb} + C_r R_{ren} + C_{ret} R_{ret} + C_q E(L_q) +$$

$$+ c(\mu C_{s1} + \nu C_{s2}) + c(\mu + \nu)(1-\beta) C_{s-f} + c C_a,$$

where (P_b), (P_{wv}), (P_{id}), (B_r), (R_{ren}), (R_{ret}), and $(E(L_q))$, are the probabilities that the servers are on normal busy period, working vacation period, idle during busy period, average balking, reneging and retention rates, and average queue length respectively. The cost elements associated C_i are defined in [4]. The objective is to determine the optimal service rate during normal busy period, μ^* using quadratic fit search method (QFSM). The cost minimization problem can be given as $\min_\mu \Gamma(\mu)$. For the numerical purpose we put $\vartheta_n = 1 - \dfrac{n}{N}$, fixe $C_b = 1,\ C_{wv} = 0.5,\ C_q = 1.5,\ C_{Rb} = 1,\ C_{ren} = 1,\ C_{id} = 0.5,\ C_{ret} = 1,\ C_{s1} = 2.5,\ C_{s2} = 2,\ C_{s-f} = 1$, and $C_a = 0.5$ and consider the following cases:

- Table 1 and Figure 1: $\lambda = 0.8$, $\beta = 0.7$, $c = 3$, $\theta = 0.4$, $\xi = 0.5$, $\alpha = 0.5$, and $N = 20$.

- Table 2 and Figure 2: $\lambda = 0.8$, $\beta = 0.7$, $c = 2$, $\nu = 0.2$, $\xi = 0.8$, $\alpha = 0.5$, and $N = 20$.
- Table 3 and Figure 3: $\lambda = 0.8$, $\beta = 0.7$, $c = 3$, $\theta = 0.4$, $\nu = 0.3$, $\alpha = 0.5$, and $N = 20$.
- Table 4 and Figure 4: $\lambda = 0.8$, $\beta = 0.7$, $\nu = 0.3$, $\theta = 0.4$, $\xi = 0.5$, $\alpha = 0.5$, and $N = 20$.

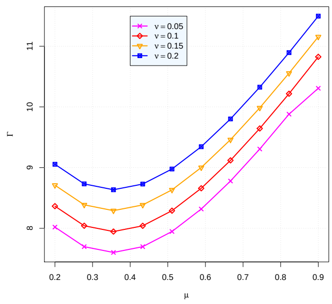

FIGURE 1. μ^* vs. Γ under SWV policy.

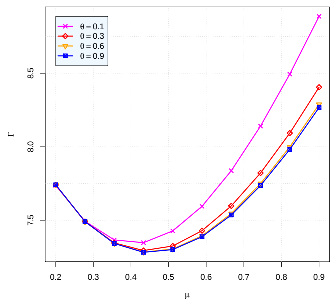

FIGURE 2. μ^* vs. Γ under SWV policy.

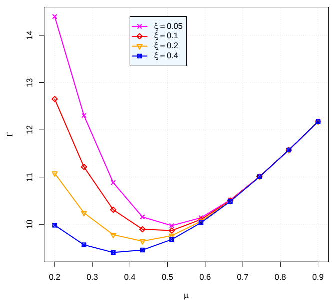

FIGURE 3. μ^* vs. Γ under MWV policy.

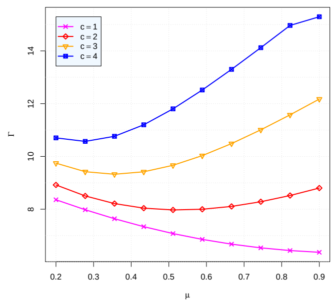

FIGURE 4. μ^* vs. Γ under MWV policy.

	$\nu = 0.05$		$\nu = 0.1$		$\nu = 0.15$	
	SWV	MWV	SWV	MWV	SWV	MWV
μ^*	0.3530611	0.3529851	0.3532569	0.3531987	0.3534304	0.3533881
Γ^*	7.602	7.602198	7.946371	7.94651	8.29081	8.290895

TABLE 1. μ^* and Γ^*, for different values of ν, under SWV and MWV policies.

	$\theta = 0.1$		$\theta = 0.3$		$\theta = 0.9$	
	SWV	MWV	SWV	MWV	SWV	MWV
μ^*	0.4082898	0.4075521	0.4418178	0.4411087	0.4526102	0.4525709
Γ^*	7.342124	7.343256	7.293472	7.294413	7.279612	7.279648

TABLE 2. μ^* and Γ^*, for different values of θ, under SWV and MWV policies.

	$\xi = 0.1$		$\xi = 0.2$		$\xi = 0.4$	
	SWV	MWV	SWV	MWV	SWV	MWV
μ^*	0.4777952	0.4777702	0.4325604	0.4325421	0.3733920	0.3733847
Γ^*	9.842556	9.842644	9.644805	9.644819	9.400941	9.400905

TABLE 3. μ^* and Γ^*, for different values of ξ, under SWV and MWV policies.

	c=2		c=3		c=4	
	SWV	MWV	SWV	MWV	SWV	MWV
μ^*	0.527004	0.527511	0.353839	0.353835	0.267373	0.267376
Γ^*	7.969997	7.969291	9.324464	9.324418	10.56634	10.56633

TABLE 4. μ^* and Γ^*, for different values of c, under SWV and MWV policies.

– Using QFSM, the optimal values of μ and the minimum expected cost $\Gamma(\mu^*)$ are shown in Tables 1-4, for different values of ν, θ, ξ, and c respectively. From Figures 1-4, it is well observed the convexity of the curves for different values of ν, θ, ξ, and c. This proves that there exists a certain value of the service rate μ that minimizes the total expected cost function for the chosen set of system parameters.

– From Tables 1-3, we observe that for different values of ν, θ, and ξ, the minimum expected cost $\Gamma(\mu^*)$ in SWV model is lower than that in MWV model, as intuitively expected. While from Table 4, $\Gamma(\mu^*)$ in SWV model is larger than that in MWV model. This can be explained by the fact that for $c = 2, 4$, the optimum service rate μ^* under SWV policy is smaller than μ^* under MWV policy. In addition, this can be because of the choice of the system parameters.

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PROCESSUS MARKOV GENERALISE ESPACE HILBERT

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23 mars 2021

Résumé

In the work that is presented, we propose methods of estimation and forecasting ona time interval by cutting a continuous time process into pieces of contiguous curves .Examples can be drawn from medical signals (ECG, EEG, EMG, ...) of financial data, meteorological data, ... this, we use functional processes that are exploited with ARB models (autoregressive process with values in a Banach space). On the inferential plane, we adopt a generalized Markov modelization to make the estimation and theprediction. The generalized Markov processes are long memory Markov processes, they can be, among others, solution of differential stochastic differential equation. Statistical techniques of these processes need to be developed to describe these processes and to apply forecasting techniques

Partially observed optimal control problem for McKean-Vlasov type EDSs

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Abstract

The paper studies partially observed optimal control problems of general McKean-Vlasov differential equations, in which the coefficients depend on the state of the solution process as well as of its law and the control variable. By applying Girsanovs theorem with a standard variational technique, we establish a stochastic maximum principle on the assumption that the control domain is convex. As an application, partially observed linear-quadratic control problem is discussed.

Keywords: Stochastic maximum principle, Partially observed optimal control, McKean-Vlasov differential equations, Probability measure.

PERIODIC INTEGER-VALUED $AR(p)$ PROCESS FOR MODELING AND FORECASTING SEASONAL COUNTS PHENOMENA.

SADOUN MOHAMED AND BENTARZI MOHAMED

ABSTRACT. This contribution proposes a periodic integer-valued autoregressif $PINAR(p)$ model, in order to analyze the number of certain arrivals in a fixed time interval with seasonal behavior. Two methods of parameters estimation will be proposed, namely : the conditional least squares (CLS) and the conditional maximum likelihood (CML) methods. Moreover, the prediction function of the model will be given using some representation of the conditional expectation. The performance of the obtained estimators, will be shown via an intensive simulation study. To assess the fitting and forecasting quality of the model, an application on two real data set will be realized to model the number of hospital admissions per month caused by influenza, and the daily counts of daytime road accidents.

2010 MATHEMATICS SUBJECT CLASSIFICATION.62F12, 62M10.

KEYWORDS AND PHRASES. Periodically correlated integer-valued process, periodic $INAR(p)$ model, conditional least squares (CLS) estimation, conditional maximum likelihood (CML) estimation.

1. DEFINE THE PROBLEM

The periodic integer-valued autoregressive ($PINAR$) model have been introduced to model counting phenomena that evolve over time with a seasonal structure. The distribution of a parametric $PINAR(p)$ process is mainly described by two blocs of parameters, namely a periodic vector auto-regression coefficient and a periodic probability distribution on positive integers belonging to parametric family, called an innovation distribution. It is worth mentioning that the applications of the periodic version of the $INAR(p)$ process are rare in the literature. In this context we can mention the work of Moriña *et al* (2011) who suggested for *the number of arrivals per week to the emergency service of the hospital in Barcelona caused by influenza* time series, an particular $INAR(2)$ process with a seasonal structure. The present paper suggests a periodic $INAR(p)$ model based on binomial thinning operator, and driven by a periodic sequence of independent random variables with some discrete distribution, to describe and forecast the seasonal time series of count. Briefly, a periodically correlated, in the sense of Gladyshev (1963) with period S (where S is a strictly positive integer, $S \geq 2$), integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$, is said to be a Periodic p-order Integer-Valued Autoregressive ($PINAR_S(p)$) model, if it is a solution of the following non-linear difference stochastic equation :

$$y_t = \sum_{i=1}^{p} \varphi_{t,i} \circ y_{t-i} + \varepsilon_t,\ t \in \mathbb{Z}, \qquad (1.1a)$$

where the underlying non-negative integer-valued process $\{y_t, t \in \mathbb{Z}\}$, is a periodically correlated, with the positive integer period S $(S \geq 2)$ and the innovation process, $\{\varepsilon_t, t \in \mathbb{Z}\}$, is a periodic sequence of independent non-negative integer-valued random variables, with some discrete distribution belonging to the parametric family $\left\{\mathbb{G}_{\underline{\alpha}_t} \left| \underline{\alpha}_t = (\alpha_{t,1}, ..., \alpha_{t,q})' \in A \subset \mathbb{R}_+^q \right.\right\}$, where A is an open, convex subset of $\mathbb{R}_+^q$. The column vector parameters $\underline{\varphi}_t = \left(\varphi_{t,1}, ..., \varphi_{t,p}\right)'$ and $\underline{\alpha}_t$ are periodic, with respect to t, with period S $(S \geq 2)$, where S is the smallest positive integer such that $\underline{\varphi}_{t+rS} = \underline{\varphi}_t$ and $\underline{\alpha}_{t+rS} = \underline{\alpha}_t$. Finally the symbol " $\circ$ " stands the binomial thinning operator proposed by Steutel and Van Harn (1979), which is defined as follows:

$$\varphi_{t,i} \circ y_{t-i} = \begin{cases} \sum_{k=1}^{y_{t-i}} Y_{k,t,i}, & if \ y_{t-i} > 0, \\ 0, & if \ y_{t-i} = 0. \end{cases} \tag{1.1b}$$

Note that the sequences of $(i.i.d)$ random variables of counts $\{Y_{k,t,i}, k \in \mathbb{N}\}$ are mutually independents for $t \in \mathbb{Z}, i = 1, \ldots, p$. $\{Y_{k,t,i}\}_{k\in\mathbb{N},t\in\mathbb{Z},i=1,\ldots,p}$ are Bernoulli variables with periodic success probability $\varphi_{s,i} \in (0,1)$, $s = 1, 2, \ldots, S$ and $i = 1, \ldots, p$, which are independent of the innovation process. So, we define the $p+q-$column vector $\underline{\theta}_s = \left(\underline{\varphi}_s'; \underline{\alpha}_s'\right)' \in (0,1)^p \times A \subset \mathbb{R}_+^p \times \mathbb{R}_+^q$, $s = 1, 2, \ldots, S$, in order to define the global vector of the parameters of the model (1.1) of dimension $(p+q)\,S$, $\underline{\theta} = \left(\underline{\theta}_1';\ \underline{\theta}_2';\ ...;\underline{\theta}_S'\right)'$.

Estimation and prediction of the model.

Let $\underline{y}^{(n)} = \left(y_1^{(n)}, \ldots, y_n^{(n)}\right)$ be a realization of a finite size n of a periodically correlated integer-valued autoregressive process $\{y_t, t \in \mathbb{Z}\}$ satisfying the periodically stationary integer-valued autoregressive model (1.1a). For the simplicity reasons, we suppose that $n = mS$, $m \in \mathbb{N}^*$ and let $t = s + \tau S, s = 1, \ldots, S$, and $\tau = 0, 1, \ldots, m-1$. We know that the CLS estimator is a $\sqrt{m}-$consistent estimator of $\left(\underline{\varphi}_s'; \mu_{\mathbb{G}_{\underline{\alpha}_s}}\right)'$ (see, e.g Du and Li, (1991)), which implies in a second step a $\sqrt{n}-$consistent estimator of $\underline{\theta}$.

Proposition 1. (*Constructing a $\sqrt{m}$-Consistent Estimator for $\left(\underline{\varphi}_s'; \mu_{\mathbb{G}_{\underline{\alpha}_s}}\right)'$*). *Let $\underline{\varphi}_s \in [0,1]^p$, υ probability measure on $\mathbb{Z}_+$ with finite support, and $\mathbb{G}_{\underline{\alpha}_s}$ such that $\mathbb{E}_{\mathbb{G}_{\underline{\alpha}_s}}[\varepsilon_0]^3 < \infty$ and $g_{\underline{\alpha}_s}(0) \in (0,1)$ then :*
$\left(\sqrt{m}\left(\overline{\underline{\varphi}}_{s,m} - \underline{\varphi}_s\right)'; \sqrt{m}\left(\overline{\mu}_{\underline{\alpha}_s,m} - \mu_{\mathbb{G}_{\underline{\alpha}_s}}\right)'\right)'$ *converges in distribution under $H_g^{(n)}(\underline{\theta})$ where :*

$$\begin{pmatrix} \overline{\underline{\varphi}}_{s,m} \\ \overline{\mu}_{\mathbb{G}_{\underline{\alpha}_s},m} \end{pmatrix} =$$

$$\begin{pmatrix} \sum_{\tau=0}^{m-1} y_{(s-1)+\tau S}^2 & \sum_{\tau=0}^{m-1} y_{(s-1)+\tau S} y_{(s-2)+\tau S} & \cdots & \sum_{\tau=0}^{m-1} y_{(s-1)+\tau S} \\ \sum_{\tau=0}^{m-1} y_{(s-2)+\tau S} y_{(s-1)+\tau S} & \sum_{\tau=0}^{m-1} y_{(s-2)+\tau S}^2 & \cdots & \sum_{\tau=0}^{m-1} y_{(s-2)+\tau S} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{\tau=0}^{m-1} y_{(s-1)+\tau S} & \sum_{\tau=0}^{m-1} y_{(s-2)+\tau S} & \cdots & m \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} \sum_{\tau=0}^{m-1} y_{(s-1)+\tau S} y_{s+\tau S} \\ \sum_{\tau=0}^{m-1} y_{(s-2)+\tau S} y_{s+\tau S} \\ \vdots \\ \sum_{\tau=0}^{m-1} y_{s+\tau S} \end{pmatrix}$$

Proposition 2. (*Conditional Maximum Likelihood Estimator*). *We call an estimator* $\left(\widehat{\underline{\theta}}_n\right)_{n\in\mathbb{Z}_+}$ *of* $\underline{\theta}$ *a conditional maximum likelihood CML estimator of* $\underline{\theta}$ *if* $\widehat{\underline{\theta}}_n$ *maximizes the conditional likelihood function associated to the model* (2.4.a), *i.e.,*

$$\forall n \in \mathbb{Z}_+ : \left(\widehat{\underline{\theta}}_n\right) \in \underset{(\underline{\theta})\in[0,1]^{pS}\times A^S}{\arg\max} \left(\prod_{\tau=0}^{m-1}\prod_{s=1}^{S} P^{\underline{\theta}_s}_{\left(y_{s-1+\tau S},\ldots,y_{s-p+\tau S}\right),y_{s+\tau S}}\right),$$

or more precisely for any $s \in \{1,\ldots,S\}$ *being fixed,*

$$\forall m \in \mathbb{Z}_+ : \left(\widehat{\underline{\theta}}_{s,m}\right) \in \underset{(\underline{\theta}_s)\in[0,1]\times A}{\arg\max} \left(\prod_{\tau=0}^{m-1} P^{\underline{\theta}_s}_{\left(y_{s-1+\tau S},\ldots,y_{s-p+\tau S}\right),y_{s+\tau S}}\right).$$

$\widehat{\underline{\theta}}_{s,m} = \left(\widehat{\underline{\varphi}}'_{s,m};\ \widehat{\underline{\alpha}}'_{s,m}\right)'$ *maximizes the sth likelihood if only if the following conditions hold :*

a) $\widehat{g}_{\underline{\alpha}_s,m}(e) = 0$ *for* $e < 0$ *and* $e > u_{s+}$, *with* $u_{s+} = \max_{\tau=0,\ldots,m-1}(y_{s+\tau S})$

b) $\widehat{\underline{\theta}}_{s,m}$ *is a solution to the (constrained) optimization problem*

$$\max_{\substack{x_{s,1},\ldots,x_{s,p}\\ z_{s,1},\ldots,z_{sq}}} \left\{\prod_{\tau=0}^{m-1}\left(\sum_{e=0}^{y_{s+\tau S}} g_{\underline{z}_s}(e) \sum_{\substack{0\le k_l\le y_{s-l+\tau S},l=1,\ldots,p\\ k_1+\ldots+k_p=y_{s+\tau S}-e}} \left[\prod_{l=1}^{p}\binom{y_{(s-l)+\tau S}}{k_l} x_{s,l}^{k_l}(1-x_{s,l})^{y_{(s-l)+\tau S}-k_l}\right]\right)\right\}$$

subject to

$0 \le x_{s,i} \le 1$ *for* $s \in \{1,\ldots,S\}$ *fixed and* $i = 1,\ldots,p$,
$g_{z_{s,j}} \ge 0$ *for* $s \in \{1,\ldots,S\}$ *fixed and* $\forall j = 1,\ldots,q$ *with* $\sum_{e=0}^{u_{s+}} g_{z_{s,j}}(e) = 1$.

We stress that we, nowhere, impose that such a maximum location is unique.

Let $\mathcal{F}_n = \sigma(y_1,\ldots,y_n)$ be the σ-algebra generated from $y_1, y_2, \ldots, y_n$, then the minimum variance predictor $y_n(1)$ of y_{n+1} is given by

$$\widehat{y}_n(1) = \mathbb{E}_{\underline{\theta}}(y_{n+1}|\mathcal{F}_n) = \sum_{i=1}^{p}\varphi_{1,i}y_{n+1-i} + \mu_{\mathbb{G}_{\underline{\alpha}_1}},$$

we can see that for $h = 2$, the minimum variance predictor $y_n(2)$ of y_{n+2} is given by

$$\widehat{y}_n(2) = \mathbb{E}_{\underline{\theta}}(y_{n+2}|\mathcal{F}_n) = \sum_{i=1}^{p}\mathbb{E}_{\underline{\theta}}\left(\varphi_{2,i}\circ y_{n+2-i}|\mathcal{F}_n\right) + \mu_{\mathbb{G}_{\underline{\alpha}_2}},$$

and according to (Yan, 1985), we have

$$\begin{aligned}\mathbb{E}_{\underline{\theta}}\left(\varphi_{2,i}\circ y_{n+2-i}|\mathcal{F}_n\right) &= \mathbb{E}_{\underline{\theta}}\left(\mathbb{E}_{\underline{\theta}}\left(\varphi_{2,i}\circ y_{n+2-i}|y_{n+2-i},\mathcal{F}_n\right)|\mathcal{F}_n\right)\\ &= \varphi_{2,i}\mathbb{E}_{\underline{\theta}}(y_{n+2-i}|\mathcal{F}_n) = \varphi_{2,i}\widehat{y}_{n+2-i} = \varphi_{2,i}\widehat{y}_n(2-i)\end{aligned}$$

by induction we can give the general formula for a $h > 2$ taking into account the periodicity of the parameters of the model, under the proposition below.

Proposition 3. (*Short-term predictor of minimum variance*). *Let* $\mathcal{F}_n = \sigma(y_1,\ldots,y_n)$ *be the* σ*-algebra generated from* $y_1,\ldots,y_n$, *then the minimum variance predictor* $y_n(h)$ *of* y_{n+h} *is given for* $h > 2$ *by*

$$\widehat{y}_n(h) = \mathbb{E}_{\underline{\theta}}(y_{n+h}|\mathcal{F}_n) = \sum_{i=1}^{p}\varphi_{r+S,i}\widehat{y}_n(h-i) + \mu_{\mathbb{G}_{\underline{\alpha}_{r+S}}},$$

where r *is the remainder of the Euclidean division of* h *over* S *i.e.,* h *and* r *are congruent modulo* S *and we write* $h \equiv r\,[S]$.

Numerical illustration.

We have evaluated the Conditional Least-Squares (CLS), and the Conditional Maximum-Likelihood (CML) estimations, on a time series, of small, moderate, and relatively large sizes ($n = 80, 400, 1000$), generated from a $PINAR_S\left(p\right)$ model driven by a periodic Negative-Binomial innovation process, $\mathcal{NB}\left(\alpha_{s,1}, \alpha_{s,2}\right)$, $s = 1, 2, 3, 4$. The true parameter values of this model are :

$$\text{Model} : \underline{\theta} = \left[\left(\varphi_1; \alpha_{1,1}, \alpha_{1,2}\right); \left(\varphi_2; \alpha_{2,1}, \alpha_{2,2}\right), \left(\varphi_3; \alpha_{3,1}, \alpha_{3,2}\right), \left(\varphi_4; \alpha_{4,1}, \alpha_{4,2}\right)\right]',$$
$$= \left[\left(0.90, 3, \exp\left(-3\right)\right); \left(0.40, 1, \exp\left(-5\right)\right); \left(0.66, 2, \exp\left(-2\right)\right); \left(0.53, 4, \exp\left(-4\right)\right)\right]'.$$

Our main goal is, on one side, to show empirically the consistency property of the CLS and CML estimators, while using Root Mean Square Error $RMSE$ criterion, and on the other side, to show empirically the CML performance over the CLS estimations.

Table 1. *Simulation results of the CLS and the CML estimates for Model*

	s	φ_s	$\widehat{\varphi}_s$	$RMSE_s$	$\alpha_{s,1}$	$\widehat{\alpha}_{s,1}$	$RMSE_s$	$\alpha_{s,2}$	$\widehat{\alpha}_{s,2}$	$RMSE_s$
CLS $n=$ 80	1	.90	.9110	.0304	3	3.4116	1.6122	.0498	.0602	.0378
	2	.40	.3681	.0941	1	1.2179	.5398	.0067	.0077	.0021
	3	.66	.6604	.0071	2	1.6414	.7309	.1353	.0918	.0469
	4	.53	.5082	.0852	4	4.3597	.8733	.0183	.0196	.0035
CML $n=$ 80	1	.90	.8983	.0200	3	3.1842	.7273	.0498	.0522	.0083
	2	.40	.3716	.0916	1	1.1965	.5343	.0067	.0076	.0020
	3	.66	.6607	.0080	2	1.6376	.6500	.1353	.0915	.0424
	4	.53	.5186	.0867	4	4.2902	.9996	.0183	.0194	.0030
CLS $n=$ 400	1	.90	.9015	.0107	3	2.9312	.3321	.0498	.0488	.0036
	2	.40	.3955	.0450	1	1.0334	.2282	.0067	.0069	.0009
	3	.66	.6602	.0029	2	1.8610	.5369	.1353	.1098	.0184
	4	.53	.5249	.0366	4	4.0495	.3291	.0183	.0185	.0013
CML $n=$ 400	1	.90	.8992	.0083	3	2.9823	.2674	.0498	.0487	.0029
	2	.40	.3928	.0409	1	1.0431	.2052	.0067	.0069	.0007
	3	.66	.6601	.0031	2	1.7777	.5312	.1353	.1115	.0119
	4	.53	.5275	.0332	4	4.0670	.3037	.0183	.0186	.0010
CLS $n=$ 1000	1	.90	.8995	.0056	3	2.9353	.2018	.0498	.0489	.0023
	2	.40	.3977	.0264	1	1.0213	.1341	.0067	.0068	.0005
	3	.66	.6598	.0018	2	1.8777	.3456	.1353	.1142	.0069
	4	.53	.5286	.0196	4	4.0233	.2204	.0183	.0184	.0008
CML $n=$ 1000	1	.90	.8998	.0036	3	2.9876	.1959	.0498	.0493	.0018
	2	.40	.3988	.0214	1	1.0079	.1436	.0067	.0068	.0006
	3	.66	.6596	.0021	2	1.8733	.4300	.1353	.1259	.0079
	4	.53	.5284	.0213	4	3.9998	.2151	.0183	.0183	.0008

Hospital admissions data. We consider first the seasonal data set of 424 observations, consisting of the weekly numbers of those diagnosed with flu in the Region of Catalonia (Spain) between 2009 and 2016. We are interested in modeling this seasonal real data by a periodic integer-valued autoregressive of order 2 and with period $S = 13$, $PINAR_{13}(2)$ model with marginal Geometric distribution. The choise of $S = 13$ instead of $S = 12$ allowed us to have more reliable results. The CLS and the CML estimates $\widehat{\underline{\theta}}_{s,cls}$ and $\widehat{\underline{\theta}}_{s,cml}$, respectively, of the parameters $\varphi_{s,i}$ and $e^{-\alpha_s} = p_s$, $s = 1, 2, ..., 13$, and $i = 1, 2$, as well as their empirical Root Mean Square Error ($RMSE$), in parentheses, are given in Table 2.

Table 2. The estimated parameters from Geometric $PINAR_{13}(2)$ Model

$s\,/P$	1	2	3	4	5	6	7	8
$\widehat{\varphi}_{s,1,cls}$	.7324	.7133	.6531	.4553	.0813	.2254	.2221	.7183
$RMSE_s$	(.6844)	(.5728)	(.2352)	(.2395)	(.0103)	(.2596)	(.2551)	(.2574)
$\widehat{\varphi}_{s,2,cls}$	.7528	.7999	.7547	.6787	.0101	.0219	.0519	.3340
$RMSE_s$	(.5120)	(.2142)	(.2185)	(.2893)	(.0103)	(.0297)	(.0718)	(.4063)
$\widehat{p}_{s,cls}$	.0029	.0006	.0006	.0009	.0370	.0605	.0800	.0292
$RMSE_s$	(.0031)	(.0007)	(.0008)	(.0012)	(.0506)	(.0772)	(.1049)	(.0367)
$\widehat{\varphi}_{s,1,cml}$	.8898	.8895	.7145	.5152	.0821	.2345	.2500	.8433
$RMSE_s$	(.6014)	(.3386)	(.2319)	(.1966)	(.0070)	(.2377)	(.2244)	(.2387)
$\widehat{\varphi}_{s,2,cml}$	.9040	.9515	.9043	.7916	.0089	.0223	.0482	.3044
$RMSE_s$	(.5051)	(.1725)	(.2038)	(.2040)	(.0064)	(.0188)	(.0448)	(.3184)
$\widehat{p}_{s,cml}$	.0046	.0010	.0012	.0016	.0643	.0774	.1233	.0462
$RMSE_s$	(.0026)	(.0006)	(.0008)	(.0010)	(.0045)	(.0522)	(.1019)	(.0116)

$s\,/P$	9	10	11	12	13
$\widehat{\varphi}_{s,1,cls}$	.7892	.6834	.7158	.4523	.6147
$RMSE_s$	(.1262)	(.2867)	(.2846)	(.2252)	(.3121)
$\widehat{\varphi}_{s,2,cls}$	.7824	.7227	.7202	.7616	.6182
$RMSE_s$	(.2193)	(.3070)	(.2728)	(.2545)	(.2285)
$\widehat{p}_{s,cls}$	.1594	.0493	.0126	.0235	.0625
$RMSE_s$	(.1575)	(.0615)	(.0163)	(.0324)	(.0623)
$\widehat{\varphi}_{s,1,cml}$	.8963	.8007	.8328	.5081	.5696
$RMSE_s$	(.0828)	(.2378)	(.2430)	(.1831)	(.2740)
$\widehat{\varphi}_{s,2,cml}$	.7964	.8795	.9214	.8778	.5609
$RMSE_s$	(.1476)	(.1872)	(.2086)	(.1884)	(.2183)
$\widehat{p}_{s,cml}$	.2671	.0827	.0192	.0416	.0391
$RMSE_s$	(.0095)	(.0288)	(.0160)	(.0255)	(.0078)

Daytime road accidents data. We consider secondly the seasonal data set of 365 observations, consisting of the daily counts of daytime road accidents in Schiphol area, in Netherlands for the year (2001). We are interested to model this seasonal real data by a periodic integer-valued autoregressive of order 2 and with period $S = 7$, $PINAR_7(2)$ with marginal Geometric distribution. The CLS and the CML estimates $\widehat{\underline{\theta}}_{s,cls}$ and $\widehat{\underline{\theta}}_{s,cml}$, respectively, of the parameters $\varphi_{s,i}$ and $e^{-\alpha_s} = p_s$, $s = 1, 2, ..., 7$, and $i = 1, 2$, as well as their empirical Root Mean Square Error ($RMSE$), in parentheses, are given in Table 3.

Table 3. The estimated parameters from Geometric $PINAR_7(2)$ Model

$s\,/P$	1	2	3	4	5	6	7
$\widehat{\varphi}_{s,1,cls}$	.1941	.1204	.2010	.2601	.1954	.1399	.1836
$RMSE_s$	(.1125)	(.1392)	(.1371)	(.1318)	(.0934)	(.1229)	(.1366)
$\widehat{\varphi}_{s,2,cls}$	.1371	.1198	.3183	.1474	2880	.1415	.2397
$RMSE_s$	(.1651)	(.1138)	(.1348)	(.1457)	(.0836)	(.1007)	(.1547)
$\widehat{p}_{s,cls}$	.1568	.1260	.1289	.1759	.9417	.2750	.1556
$RMSE_s$	(.3632)	(.4993)	(.7650)	(.9358)	(.1861)	(1.060)	(.3813)
$\widehat{\varphi}_{s,1,cml}$	.1758	.0176	.1817	.2600	.1903	.0796	.1329
$RMSE_s$	(.0280)	(.0236)	(.0375)	(.0343)	(.0190)	(.0280)	(.0471)
$\widehat{\varphi}_{s,2,cml}$	.0321	.0474	.3372	.0643	.2941	.1348	.1829
$RMSE_s$	(.0426)	(.0312)	(.0394)	(.0216)	(.0173)	(.0309)	(.0590)
$\widehat{p}_{s,cml}$	.1602	.1251	.1325	.1681	.1312	.2705	.1514
$RMSE_s$	(.3021)	(.4977)	(.4055)	(.3834)	(.2053)	(.3583)	(.2922)

Concluding comments. We have proposed, a periodic $INAR(p)$ model to provide a more flexible modeling and forecasting framework, which is able to capture the features of the data such as seasonal effects. We have also annonciated a definition and some existing results concerning the proposed model. The conditional least squares (CLS) and conditional maximum likelihood (CML) estimators are established, and the performance of the obtained estimators are studied via simulation study. Two real data examples are also illustrated to show the goodness of the fit and the prediction of our $PINAR_S(p)$ model.

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PROBLEM OF BSDE UNDER G-BROWNIAN MOTION

GUESRAYA SABRINA AND DR.CHALA ADEL

Abstract. we study the optimal control by G-Backward stochastic differential equation.we adapt the stochastic maximum principle to find necessary and sufficient conditions for the optimal control of G-BSDE.

Keywords and phrases. G-Brownian motion, stochastic maximum principle, G-Backward stochastic differential equation (G-BSDE).

1. Define the problem

For any $u \in U$ we consider the backward stochastic differential equations by a G-Brownian motion $(B_t)_{t\geq 0}$ in the following form:

$$\begin{cases} dy_t = f(t, y_t, z_t, u_t)\, dt + g(t, y_t, z_t, u_t)\, d\langle B^G \rangle_t - Z_t dB_t^G + dk_t, \\ y_T = \xi. \end{cases}$$

where

$$f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times U \longrightarrow \mathbb{R}$$
$$g : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times U \longrightarrow \mathbb{R}$$
$$K \ is \ a \ \textbf{decreasing G} - \textbf{martingale}$$

The expected cost is given by

$$J(u) = \mathbb{E}\left[\int_0^T h(t, y_t, z_t, u_t)\, dt + g(y_0)\right]$$

where

$$h : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times U \longrightarrow \mathbb{R}$$
$$g : \mathbb{R} \longrightarrow \mathbb{R}$$

let T be a fixed strictly positive real number and consider the following sets

A is a closed and convex of $\mathbb{R}$

U the class of measurable,adapted processes $u : [0,T] \times \Omega \longrightarrow A$

we shall denote by U the class of measurable,adapted processes $u \in U$ such that

$$\mathbb{E}\left[\int_0^T |\, u_t\, |^2\, dt\right] < \infty$$

Almost surely, such $u \in U$ are called **admissible control processes**.

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QUADRATIC BSDES WITH TWO REFLECTING BARRIERS AND A SQUARE INTEGRABLE TERMINAL VALUE

ROUBI ABDALLAH, LABED BOUBAKEUR, AND BAHLALI KHALED

Abstract. We consider backward stochastic differential equations (BSDEs) with two reflecting barriers which generator $H(t, \omega, y, z)$ has a quadratic growth in its z-variable and a square integrable terminal value ξ. The solutions is constrained to stay between two time continuous processes L and U (called the barriers). We establish the existence of solutions when $H(t, \omega, y, z) = f(y)|z|^2$ and also when $H(t, \omega, y, z) = a+b|y|+c|z|+f(y)|z|^2$. The uniqueness and the comparison of solutions are also established when the generator is of the form $f(y)|z|^2$. The main tools are Krylov's estimate and ItÃ´-Krylov's formula, which are proved here, for the solutions of backward stochastic differential equations with two reflecting barriers.

MSC 60H10, 60H20

Keywords and phrases. Reflected quadratic BSDE; Local time; Occupation time formula; Krylov's inequality; Itô–Krylov's formula; Tanaka's formula.

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RETRIAL QUEUEING MODEL WITH BERNOULLI FEEDBACK AND ABANDONED CUSTOMERS

RAMDANI HAYAT, AMINA ANGELIKA BOUCHENTOUF, LAHCENE YAHIAOUI, AND ABBES RABHI

ABSTRACT. In this work, we present the necessary stability condition of a retial queueing system with two orbits,abandoned and feedback customers. Two independent Poisson streams of customers arrive to the system, and flow into a single-server service system. An arriving one of type i; $i = 1; 2$, is handled by the server if it is free; otherwise, it is blocked and routed to a separate type-i retrial (orbit) queue that attempts to re-dispatch its jobs at its specific Poisson rate. The customer in the orbit either attempts service again after a random time or gives up receiving service and leaves the system after a random time. The impatient customers, via certain mechanism, can be retained in the system with some probability. In addition, the customer will decide either to join the retrial group again for another service or leave the system forever with some probability.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 60K25; 68M20; 90B22

KEYWORDS AND PHRASES. Queueing system, call center, retrial queue, abandonment, feedback.

1. INTRODUCTION

The study of retrial queues in queueing theory has attracted the attention of many authors because of their wide applicability in web access, telephone switching systems, telecommunication networks and computer networks, and many daily life situations (Artelejo [2], Arivudainambi [1], Bouchentouf and Belarbi [4], Bouchentouf et al. [5, 6], and Boualem [3]).

In this work, we consider a Markovian retrial queueing system with two classes of jobs and constant retrial, abandonment and feedback customers. Two independent Poisson streams of jobs, S_1 and S_2, flow into a single-server service system. The service system can hold at most one job. The arrival rate of stream S_i is α_i, $i; i = 1, 2$, with $\alpha_1 + \alpha_2 = \alpha$. The required service time of each job is independent of its type and is exponentially distributed with mean $\frac{1}{\mu}$. If an arriving type-i job finds the (main) server busy, it is routed to a dedicated retrial (orbit) queue from which jobs are re-transmitted at an exponential rate. The rates of retransmissions may be different from the rates of the original input streams. So, the blocked jobs of type i form a type-i single-server orbit queue that attempts to retransmit jobs (if any) to the main service system at a Poisson rate γ_i; $i = 1, 2$. This creates a system with three dependent queues. The customer in the orbit either attempts service again after a random time or gives up receiving service and leaves the system after a random time at rate δ_i, $i = 1, 2$. The impatient customers may leave the system with probability θ. Via certain mechanism, they can be retained in the system with probability $\theta' = 1 - \theta$.

After the customer is completely served , it will decide either to join the retrial group again for another service with probability β or to leave the system forever with probability $\overline{\beta} = 1 - \beta$.

2. Main Result

Let $C(t)$ denotes the number of jobs in the main queue. $C(t)$ takes the values of 0 or 1. Let $N_i(t)$ be the number of jobs in orbit queue i, $i = 1, 2$. The Markov process $(N_1(t), N_2(t), C(t)) : \{t \in [0, +\infty]\}$ is irreducible on the state-space $\{0, 1, ...\} \times \{0, 1, ...\} \times \{0, 1\}$. Such a network can serve as a model for two competing job streams in a carrier sensing multiple access system 'CSMA'. A Local Area Computer Network (LAN) can be an example of CSMA. The main goal of this work is to give the necessary stability condition of a retrial queueing system with two orbits, constant retrials, abandoned and feedback customers. The main result is given in the following proposition.

Proposition 2.1. *The following condition*

$$\frac{\alpha(\gamma_1 + \theta\delta_1)(\gamma_2 + \theta\delta_2)}{[\alpha + (\beta + 1)\mu](\gamma_1 + \theta\delta_1)(\gamma_2 + \theta\delta_2) - \alpha\gamma_1\gamma_2 - \alpha_1\theta\delta_1\gamma_2 - \alpha_2\theta\delta_2\gamma_1}$$

$$\times\left(1 + \frac{\alpha_i}{\gamma_i + \theta\delta_i}\right) < 1,$$

for $i = 1, 2$ *and*

$$[\alpha + (\beta + 1)\mu](\gamma_1 + \theta\delta_1)(\gamma_2 + \theta\delta_2) - \alpha\gamma_1\gamma_2 - \alpha_1\theta\delta_1\gamma_2 - \alpha_2\theta\delta_2\gamma_1 \neq 0$$

is necessary for the stability of the system.

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Regime switching Merton model under general discount function: Time-consistent strategies

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February 27, 2021

Abstract

In this presentation, we revisit the equilibrium consumption–investment for Merton's portfolio problem with a general discount function and a general utility function in a Markovian framework. The coefficients in our model, including the appreciation rate and volatility of the stock, are assumed to be Markov modulated processes. The investor receives a deterministic income, invests in risky assets and consumes continuously. The objective is to maximize the terminal wealth and accumulated consumption utility. The non-exponential discounting makes the optimal strategy adopted time-inconsistent. Consequently, the Bellman's optimality principle does no longer hold. We formulate the problem in the game theoretic framework and by using a variational technical approach, we derive the necessary and sufficient equilibrium condition. An closed loop form of the equilibrium found for a set of special utility functions (logarithme and power) enable us to discuss some interesting optimal investment strategies that have not been revealed before in literature.

Keys words: Investment-Consumption Problem, Merton Portfolio Problem, Equilibrium Strategies, Non-Exponential Discounting, Stochastic Optimization.

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Abstract

This work is concerned with the problem of selecting a suitable bandwidth, for the M-estimator of the robust regression function from left truncated and right censored data (LTRC), under strong mixing condition : After giving an extension of the asymptotic result of Nadaraya (1989, Theorem 1.2) into the context of robust regression estimator under dependence. We provide an asymptotic expression for the mean integrated squared error (MISE) of this estimator. As a consequence, a bandwidth selector based on iterative plug-in ideas is introduced. We also present a robust version of the Least Square Cross-Validation (RLSCV) bandwidth selection. A simulation study is investigated to examine the practical performance of both two methods.

SPDEs with space interactions - a model for optimal control of epidemics

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31 October 2020

Abstract

We consider optimal control of a new type of stochastic partial differential equations (SPDEs). The SPDEs have space interactions, in the sense that the dynamics of the system at time t and position in space x also depend on the space-mean of values at neighbouring points. This is a model with many applications, e.g. to population growth studies and epidemiology. We prove the existence and uniqueness of solutions of a class of SPDEs with space interactions, and we show that, under some conditions, the solutions are positive for all times if the initial values are. Sufficient and necessary maximum principles for the optimal control of such systems are derived. Finally, we apply the results to study an optimal vaccine strategy problem for an epidemic by modelling the population density as a space-mean stochastic reaction-diffusion equation.

Keywords: SPDE; space interactions, epidemics; optimal vaccine strategy; maximum principle.

1 Introduction

We are all faced with decisions, both our own and others. When considering decisions in mathematics, we use the theory of optimal control. As we make many decisions under uncertainty. Stochastic control theory provides us with a powerful tool to handle many cases, like how to run a production optimally with respect to economic and environmental criteria, when a factory should order new equipment, when a financial institution or an individual should buy and sell stocks in the financial market, how to find sustainable harvesting strategies in agriculture and fishing, and how to deal optimally with epidemics that we are interested in the current paper.

To be able to apply mathematical theory and methods to such problems, the situations have to be put into a mathematical context. Usually the system we consider is not static, but changes with time. This makes it natural to use dynamical systems as models.

In the present paper we will use a generalized stochastic heat equation with space interactions as a model for epidemics. By space interactions we mean that the dynamics of the population density at a point x depends not only on its value and derivatives at x, but also on the density values in a neighbourhood of x. For example, define G to be a space-averaging operator of the form

$$G(x,\varphi) = \frac{1}{V(K_\theta)} \int_{K_\theta} \varphi(x+y)dy; \quad \varphi \in L^2(\mathbb{R}^n), \tag{1.1}$$

where $V(\cdot)$ denotes Lebesgue volume and

$$K_\theta = \{y \in \mathbb{R}^n; |y| < \theta\}$$

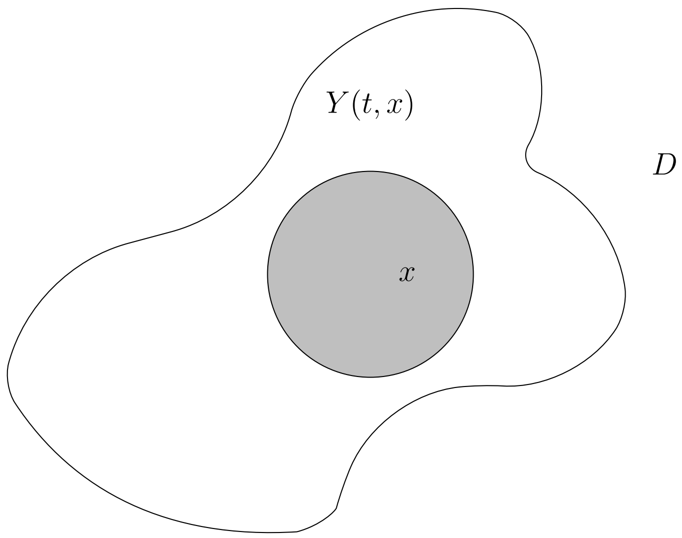
is the ball of radius $r > 0$ in $\mathbb{R}^n$ centered at 0. Then

$$\overline{Y}_G(t,x) := G(x, Y(t,\cdot))$$

is the average value of $Y(t, x+\cdot)$ in the ball K_θ.
More generally, if we are given a nonnegative measure (weight) $\rho(dy)$ of total mass 1, then the ρ-weighted average of Y at x is defined by

$$\overline{Y}_\rho(t,x) := \int_D Y(t,x+y)\rho(dy).$$

We believe that by allowing interactions between populations at different locations, we get a better model for population growth, including the modelling of epidemics. For example, we know that COVID-19 is spreading by close contact in space.



2 Solutions of SPDEs with space interactions, and positivity

Fix $t > 0$, and let $k \in \mathbb{N}_0 = \{0, 1, 2, \dots, \dots\}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) = \mathbb{N}_0^n$. For functions $f \in C^\infty(\mathbb{R}^n)$, we let

$$|f|_{k,\alpha} = \sup_{x\in D, \beta\leq\alpha} \left(1 + |x|^k\right) \left|\partial^\beta f(x)\right|; k \in \mathbb{N}_0; \beta = (\beta_1, \beta_2, ...\beta_n), \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n,$$

denote the Schwartz family of seminorms, and we let $\mathcal{S}(\mathbb{R}^n)$ be the set of f such that $|f|_{k,\alpha} < \infty$ for all k, α.

Let $\mathcal{Y}_{k,\alpha}^{(t)}$ denote the family of random fields $Y(s,x) = Y(s,x,\omega)$, such that $||Y||_{k,\alpha}^{(t)} < \infty$ where

$$\|Y\|_{k,\alpha}^{(t)} = \mathbb{E}\left[\sup_{s\leq t}\left\{|Y(s,.)|_{k,\alpha}^2\right\}\right]^{\frac{1}{2}},$$

and let $\mathcal{Y}^{(t)}$ be the intersection (projective limit) of all the spaces $\mathcal{Y}_{k,\alpha}^{(t)}; k \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n$. We can now prove the following:

Theorem 2.1 *Let $\xi \in \mathcal{S}(\mathbb{R}^n)$ and let $h : [0,T] \mapsto \mathbb{R}$ be bounded and deterministic.*

- *Then there exists a unique solution $Y(t,x) \in \mathcal{Y}^{(T)}$ of the following SPDE with space interactions*

$$Y(t,x) = \xi(x) + \int_0^t LY(s,x)ds + \int_0^t \overline{Y}(s,x)ds + \int_0^t h(s)Y(s,x)dB(s); \quad t \in [0,T].$$

- *if $\xi(x) \geq 0$ for all $x \in \mathbb{R}^n$, we have $Y(t,x) \geq 0$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$.*

3 The optimization problem

We now give a general formulation of the problem we consider.
Let $T > 0$ and we assume that the state $Y(t,x)$ at time $t \in [0,T]$ and at the point $x \in \overline{D} := D \cup \partial D$ satisfies the generalised quasilinear stochastic heat equation:

$$\begin{cases} dY(t,x) & = A_x Y(t,x)dt + b(t,x,Y(t,x),Y(t,\cdot),u(t,x))dt \\ & +\sigma(t,x,Y(t,x),Y(t,\cdot),u(t,x))dB(t), \\ Y(0,x) & = \xi(x); \quad x \in D, \\ Y(t,x) & = \eta(t,x); \quad (t,x) \in (0,T) \times \partial D. \end{cases} \tag{3.1}$$

The process $u(t,x) = u(t,x,\omega)$ is our control process, assumed to have values in a given convex set $U \subset \mathbb{R}^k$. We assume that $u(t,x)$ is $\mathbb{F}$-predictable for all $(t,x) \in (0,T) \times D$.

We call the control process $u(t,x)$ admissible if the corresponding SPDE with space-mean dynamics (3.1) has a unique strong solution $Y \in \mathcal{Y}^T$ with values in a given set $\mathbf{S} \subset \mathbb{R}$. The set of admissible controls is denoted by $\mathcal{U}$.
The performance functional (cost) associated to the control u is assumed to have the form

$$J(u) = \mathbb{E}\Big[\int_0^T \int_D f(t,x,Y(t,x),Y(t,\cdot),u(t,x))dxdt + \int_D g(x,Y(T,x),Y(T,\cdot))dx\Big]; \quad u \in \mathcal{U}. \tag{3.2}$$

Problem 3.1 *Find $\widehat{u} \in \mathcal{U}$ such that*

$$J(\widehat{u}) = \inf_{u\in\mathcal{U}} J(u). \tag{3.3}$$

$$\begin{aligned} H(t,x,y,\varphi,u,p,q) := H(t,x,y,\varphi,u,p,q,\omega) = f(t,x,y,\varphi,u) + b(t,x,y,\varphi,u)p \\ + \sigma(t,x,y,\varphi,u)q. \end{aligned} \tag{3.4}$$

We associate to the Hamiltonian the following backward SPDE

$$dp(t,x) = -\left[A_x^* p(t,x) + \tfrac{\partial H}{\partial y}(t,x) + \overline{\nabla}_\varphi^* H(t,x)\right] dt + q(t,x)dB(t), \tag{3.5}$$

with boundary/terminal values

$$\begin{cases} p(T,x) & = \frac{\partial g}{\partial y}(x) + \overline{\nabla}_\varphi^* g(x); \quad x \in D, \\ p(t,x) & = 0; \quad (t,x) \in (0,T) \times \partial D. \end{cases} \tag{3.6}$$

Theorem 3.2 (Sufficient Maximum Principle) *Suppose $\widehat{u} \in \mathcal{U}$, with corresponding $\widehat{Y}(t,x), \widehat{p}(t,x), \widehat{q}(t,x)$. Suppose the functions $(y,\varphi) \mapsto g(x,y,\varphi)$ and $(y,\varphi,u) \mapsto H(t,x,y,\varphi,u,\widehat{p}(t,x),\widehat{q}(t,x))$ are convex for each $(t,x) \in [0,T] \times D$. Moreover, suppose that, for all $(t,x) \in [0,T] \times D$,*

$$\begin{aligned} &\min_{v\in U} H(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),v,\widehat{p}(t,x),\widehat{q}(t,x)) \\ &= H(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),\widehat{u}(t,x),\widehat{p}(t,x),\widehat{q}(t,x)). \end{aligned}$$

Then $\widehat{u}$ is an optimal control.

We now go to the other version of the necessary maximum principle which can be seen as an extension of Pontryagin's maximum principle to SPDE with space-mean dynamics. Here concavity assumptions are not required . We consider the following:
Given arbitrary controls $u, \widehat{u} \in \mathcal{U}$ with u bounded, we define

$$u^\theta := \widehat{u} + \theta u; \quad \theta \in [0,1].$$

Note that, thanks to the convexity of U, we also have $u^\theta \in \mathcal{U}$. We denote by $Y^\theta := Y^{u^\theta}$ and by $\widehat{Y} := Y^{\widehat{u}}$ the solution processes of (3.1) corresponding to u^θ and $\widehat{u}$, respectively.

Define the derivative process $Z(t,x)$ by the following equation, which is obtained by differentiating $Y^\theta(t,x)$ with respect to θ at $\theta = 0$:

$$\begin{cases} dZ(t,x) &= \left\{ A_x Z(t,x) + \frac{\partial b}{\partial y}(t,x)Z(t,x) + \langle \nabla_\varphi b(t,x), Z(t,\cdot)\rangle + \frac{\partial b}{\partial u}(t,x)u(t,x) \right\} dt \\ &\quad + \left\{ \frac{\partial \sigma}{\partial y}(t,x)Z(t,x) + \langle \nabla_\varphi \sigma(t,x), Z(t,\cdot)\rangle + \frac{\partial \sigma}{\partial u}(t,x)u(t,x) \right\} dB(t), \\ Z(t,x) &= 0; \quad (t,x) \in (0,T) \times \partial D, \\ Z(0,x) &= 0; \quad x \in D. \end{cases} \tag{3.7}$$

Theorem 3.3 (Necessary Maximum Principle) *Let $\widehat{u}(t,x)$ be an optimal control and $\widehat{Y}(t,x)$ the corresponding trajectory and adjoint processes $(\widehat{p}(t,x), \widehat{q}(t,x))$. Then we have*

$$\left.\frac{\partial \widehat{H}}{\partial u}\right|_{u=\hat{u}}(t,x) = 0; \qquad a.s.$$

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SAMPLE SIZE CALCULATIONS IN PHASE II CLINICAL TRIALS USING THE PREDICTION OF SATISFACTION DESIGN.

ZOHRA DJERIDI AND HAYET MERABET

ABSTRACT. Djeridi and Merabet [2] proposed a hybrid frequentist-Bayesian approach to phase II clinical trials with binary outcomes and continuous monitoring. The efficacy of an experimental treatment E is evaluated based on data from an uncontrolled trial of E. The trial continues until E is shown with high prediction of satisfaction to be promising or not promising, or until a predetermined maximum sample size is reached. In this paper, we study the design structure, describe sample size and monitoring criteria and provide numerical guidelines for implementation. We also examine the effects of intermittent monitoring on the design's properties. This study gives criteria from early termination of trials unlikely to yield conclusive results, based on the predictive distribution of the remaining interim analysis to evaluate the chance to continue the trial til its term.

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SENSITIVITÉ DES PERFORMANCES DE L'ESTIMATEUR À NOYAU D'UNE DENSITÉ CONDITIONNELLE AU CHOIX DU PARAMÈTRE DE LISSAGE

LADAOURI NOUR EL HAYET AND CHERFAOUI MOULOUD

ABSTRACT. Dans ce travail, nous nous sommes intéressés à l'analyse de l'impact du choix du paramètre de lissage sur les performances (ISE moyenne et le temps des calculs) de l'estimateur à noyau d'une densité conditionnelle, $f(y/x)$. Plus précisément, nous avons considéré le choix du paramètre de lissage par la minimisation de l'ISE sous deux différentes hypothèses, à savoir : H_1 : *les paramètres de lissage dans la direction de X et de Y sont indépendant* et H_2 : *le paramètre de lissage dans la direction de X et le même que celui dans la direction de Y.*

Pour ce faire, nous avons réalisé une application numérique comparative, basée sur des échantillons artificiels, sur deux exemples. Les résultats des simulations obtenus dans notre application, sur des échantillons de différentes tailles en utilisant le noyau Normal et le noyau d'Epanechnikov pour la construction de l'estimateur de $f(y/x)$, mettent en relief l'impact des deux hypothèses H_1 et H_2 sur les performances retenus.

KEYWORDS AND PHRASES. Estimation à noyau, densité conditionnelle, erreur, simulation.

1. INTRODUCTION

Soit X et Y deux variables aléatoires uni-variées de densité jointe $g(x, y)$; et m est la densité marginal de X. On considère $\{(x_1, y_1); \; . . . \; ; \; (x_n, y_n)\}$ $n-$observations issues de la variable aléatoire (X, Y). L'estimateur à noyau $\hat{f}(y/x) = \frac{\hat{g}(x,y)}{\hat{m}(x)}$ de la densité conditionnelle $f(y/x)$, introduit initialement par Rosenblatt en 1969 [7], est donné sous la forme suivante:

$$\hat{f}(y/x) = \frac{\frac{1}{nab}\sum_{j=1}^{n} K(\frac{x-x_j}{a})K(\frac{y-y_j}{b})}{\frac{1}{na}\sum_{j=1}^{n} K(\frac{x-x_j}{a})} = \frac{1}{b}\sum_{j=1}^{n} W_{j,a}(x)K\left(\frac{y-y_j}{b}\right), \tag{1}$$

avec K est un noyau sur $\mathbb{R}$, $a > 0$ est le paramètre de lissage dans la direction de X et $b > 0$ est le paramètre de lissage dans la direction de Y. De plus, afin d'assurer la convergence de cet estimateur les deux paramètres de lissage doivent vérifier les conditions: $a, b \to 0$ et $nab \to \infty$ lorsque $n \to \infty$ (pour plus de détails voir [4]).

Dans la littérature une autre version simplifiée de (1) a été proposée. En effet, sous l'hypothèse que les deux paramètres de lissage a et b, respectivement dans la direction de X et dans la direction de Y, sont les mêmes $(h = a = b)$ l'expression (1) peut être réécrite sous sa forme simplifier donnée par:

$$(2)\quad \hat{f}(y/x) = \frac{\frac{1}{nh^2}\sum_{j=1}^{n} K\left(\frac{x-x_j}{h}\right) K\left(\frac{y-y_j}{h}\right)}{\frac{1}{nh}\sum_{j=1}^{n} K\left(\frac{x-x_j}{h}\right)} = \frac{1}{h}\sum_{j=1}^{n} W_{j,h}(x) K\left(\frac{y-y_j}{h}\right),$$

où K est un noyau sur $\mathbb{R}$ et h est le paramètre de lissage. De plus, pour que l'estimateur soit convergent le paramètre de lissage doit satisfaire les conditions suivantes: $h \to 0$ et $nh^2 \to \infty$ lorsque $n \to \infty$ (pour plus de détails voir [9]).

Il est claire, d'après les deux expressions (1) et (2), que la mise en œuvre de cette technique nécessite de fixer le noyau K et le(s) paramètre(s) de lissage. Pour le noyau K, les choix plus communs du noyau sont définis en termes de fonction de densité de probabilité univariée et unimodale, ce qui coïncide avec les choix qu'on réalise dans l'estimation d'une densité classique. Tandis que pour le choix du paramètre de lissage, une petite inspection de la littérature nous permet de constater qu'il existe deux catégories de procédures de sélection, à savoir : celle où on considère l'estimateur définie dans (1) (exemple de [4, 1]) et celle où on impose l'hypothèse d'égalité des paramètres de lissage a et b ($a = b = h$) respectivement de la direction de x et de la direction de y, c'est-à-dire celle qui regroupe les technique de sélection spécifiques à l'estimateur définie dans (2) (exemple de [9]).

Dans le présent travail, nous avons proposé d'analyser et de comparer numériquement les performances des deux estimateurs de $f(y|x)$ définis par (1) et (2).

Le reste du document est organisé comme suit : Dans la section 2, nous allons aborder brièvement le problème du choix du noyau et des paramètres de lissage dans le cadre d'estimation à noyau d'une densité conditionnelle univarié. Avant de conclure dans la section 4, nous allons présenter dans la section 3 l'application numérique réalisée sur des échantillons simulés, les résultats numériques et graphiques obtenus ainsi que la discussion de ces derniers.

2. Choix du noyau et des paramètres de lissage

Le problème du choix du noyau K, que ce soit pour l'estimateur (1) ou l'estimateur (2), reste le même que dans le cas d'estimation d'une densité unimodèle. De ce fait, le choix du noyau K doit seulement être adapté au support de la densité [8, 2, 3, 6, 5].

Les paramètres de lissage optimaux peuvent être obtenus par la différentiation de l'expression du $MISE$ associée à l'estimateur par rapport à a et b dans le cas de l'estimateur (1) et par rapport à h dans le cas de l'estimateur (2) et en égalisant à zéro les dérivées obtenues.

Dans le cas de l'estimateur (1), Hyndman et al. [4] ont montré que le couple (a, b) optimal au sens du $MISE$ est la solution du système d'équations suivant:

$$(3)\quad \begin{cases} -\frac{c_1}{n} + \frac{c_2 b}{n} + 4c_3 a^5 b + 2c_5 a^3 b^3 & = \ 0; \\ -\frac{c_1}{n} + 4c_4 a b^5 + 2c_5 a^3 b^3 & = \ 0; \end{cases}$$

où $c_1,\ c_2,\ c_3,\ c_4$ et c_5 sont des constantes qui dépendent du noyau K, la densité conditionnelle $f(y/x)$ et de la densité marginal $m(x)$, et qui sont donnés par :

$$(4)\quad \begin{cases} c_1 &= \int R^2(K)dx, \\ c_2 &= \int\int R(K)f^2(y/x)dydx, \\ c_3 &= \int\int \frac{\sigma_K^4 m(x)}{4}\left\{2\frac{m'(x)}{m(x)}\frac{\partial f(y/x)}{\partial x} + \frac{\partial^2 f(y/x)}{\partial x^2}\right\}^2 dydx, \\ c_4 &= \int\int \frac{\sigma_K^4 m(x)}{4}\left\{\frac{\partial^2 f(y/x)}{\partial y^2}\right\}^2 dydx, \\ c_5 &= \int\int \frac{\sigma_K^4 m(x)}{2}\left\{2\frac{m'(x)}{m(x)}\frac{\partial f(y/x)}{\partial x} + \frac{\partial^2 f(y/x)}{\partial x^2}\right\}\left\{\frac{\partial^2 f(y/x)}{\partial y^2}\right\} dydx, \end{cases}$$

avec $R(g) = \int g^2(x)dx$ et σ_K^2 est la variance du noyau K.

De plus, ils ont montré également que la solution du système (3) est donnée par:

$$(5)\quad \begin{cases} a^* &= c_1^{1/6}\left\{4\left(\frac{c_3^5}{c_4}\right)^{1/4} + 2c_5\left(\frac{c_3}{c_4}\right)^{3/4}\right\}^{-1/6} n^{-1/6}; \\ b^* &= \left(\frac{c_3}{c_4}\right)^{1/4} a^*, \end{cases}$$

avec les constantes $c_1,\ c_2,\ c_3,\ c_4$ et c_5 sont donné dans (4).

A partir de l'expression (5), on remarque que dans le cadre pratique a^* et b^* ne sont pas exploitables et ceci le fait que la quantification de ces derniers dépendent des fonctions inconnues, f et m à travers les constantes $c_i,\ i = \overline{1,5}$. Ci-dessous les deux techniques de sélection du paramètre de lissage les plus utilisées dans la pratique.

(1) **La règle de référence:** La technique de *règle de référence*, proposée initialement par Silverman [8] dans le cadre d'estimation de densité unimodèle, vise à substitué les fonctions inconnues intervenant dans la définition du paramètre de lissage optimal par des fonctions connues afin qu'on puisse quantifier le paramètre de lissage optimal.

Dans le cadre d'estimation à noyau d'une densité conditionnelle la méthode de règle de référence associée à l'estimateur (1) a été considéré par Bashtannyk et Hyndman [1]. Les auteurs ont proposé de remplacer la densité conditionnelle $f(y/x)$ dans l'expression (4) par une densité d'une loi normale de moyenne $r(x) = u + vx$ et d'écart type $\sigma(x) = p + qx$ avec $v \neq 0$. Concernant la densité marginale $m(x)$, les auteurs ont proposé de substituer cette fonction dans (4), dans un premier temps, par une loi normale ayant une variance constante et, dans un second temps, par une loi uniforme définie sur $[\alpha, \beta]$.

A base d'une étude de simulation Bashtannyk et Hyndman [1] ont montré que cette technique est robuste et elle fournie des résultats raisonnables, même pour des densités qui ne sont pas des distributions normale.

(2) **Validation croisée:**

Le principe de cette technique est d'estimer une performance (ISE, MISE, vraisemblance,...) associée à l'estimateur considéré par le principe de validation croisée et le paramètre de lissage dans ce cas et celui qui optimise (minimise ou maximise, selon le critère considéré) l'estimateur obtenu.

Dans [9] l'auteur à considérer le choix du paramètre de lissage, associé à l'estimateur (2), par la technique de validation croisée. L'auteur a montré que si on opte pour la minimisation du critère ISE par l'approche validation croisée alors la sélection du paramètre de lissage optimal consiste à déterminer la valeur de h de minimiser le critère suivant:

(6)
$$CV(h) = \frac{1}{n}\sum_{i=1}^{n} W(X_i)\int\left(\frac{g_h^{-i}(X_i,y)}{m_h^{-i}(X_i)}\right)^2 W^{'}(y)dy - \frac{2}{n}\sum_{i=1}^{n}\frac{g_h^{-i}(X_i,Y_i)}{m_h^{-i}(X_i)}W(X_i)W^{'}(Y_i).$$

où,

$$\begin{aligned}\hat{g}^{-i}(x,y) &= \tfrac{1}{(n-1)h^2}\sum_{j\neq i}^{n} K\left(\tfrac{x-X_j}{h}\right)K\left(\tfrac{y-Y_j}{h}\right),\\ \hat{m}^{-i}(x) &= \tfrac{1}{(n-1)h}\sum_{j\neq i}^{n} K\left(\tfrac{x-X_j}{h}\right).\end{aligned}$$

3. Performances numérique de l'estimateur de $f(Y/X)$

L'objectif de la présente section est de mettre en évidence numériquement la qualité des estimations (1) et (2) au sens de l'ISE moyenne ainsi que au sens du temps moyen de calcul nécessaire pour la mise en oeuvre de ces deux estimateurs.

Afin de distinguer les deux estimateurs en question dans le reste du présent document nous allons adopter les notations $\hat{f}_{ab}$ et $\hat{f}_h$ pour désigner respectivement l'estimateur donné dans (1) et (2).

3.1. **Description des paramètres de l'application.**

Pour répondre à notre objectif nous avons implémenté un simulateur sous Matlab dont ses principales étapes sont :

(1) Générer N échantillons (X,Y) de taille n d'une loi cible.
(2) Calculer (a^*,b^*) et h^* qui minimisent les ISE moyennes associées aux deux estimateurs.
(3) Calculer $\hat{f}_{ab}$ et $\hat{f}_h$ et comparer leurs performances.

Pour réaliser ces étapes, pour des raisons calculatoire nous avons proposé de discrétiser l'ISE moyenne. Ainsi, en prenant en considération les N échantillons générés, l'expression du ISE moyenne sera approchée par:

(7)
$$AISE = \frac{\Delta}{nN}\sum_{l=1}^{N}\sum_{j=1}^{J}\sum_{i=1}^{n}\left[\hat{f}(y_j^{'}/X_i) - f(y_j^{'}/X_i)\right]^2,$$

où $y^{'} = \left(y_1^{'}, y_2^{'}, ..., y_J^{'}\right)$ est un vecteur de points équidistants dans l'espace de Y et $\Delta = y_{j+1}^{'} - y_j^{'}$, $\forall j \in \{1,2,...,J-1\}$.

Par conséquent, les estimations des paramètres de lissage optimaux au sens du ISE moyenne correspondent dans ce cas aux quantités minimisant l'expression (7).

Pour l'application numérique nous avons repris les deux exemples présentés par Bashtannyk et Hyndman dans [1] et qui sont définis comme suit:

Modèle 1:

$$y = 10 + 5X + \epsilon, \tag{8}$$

où X et ϵ sont deux variables aléatoires issues respectivement $\mathcal{N}(10, 9)$ et $\mathcal{N}(0, 100)$ avec $\mathcal{N}(\mu, \sigma^2)$ désigne une loi normale de moyenne μ et de variance σ^2. Pour ce modèle, il est facile de montré que la densité de la variable aléatoire Y sachant X est définie par:

$$f(y/x) = \frac{1}{10}\phi\left(\frac{y - 10 - 5x}{10}\right), \tag{9}$$

avec $\phi(.)$ est la densité d'une distribution normale centrée réduite.

Modèle 2:

$$y = 2\sin(\pi X) + \epsilon, \tag{10}$$

où X et ϵ sont deux variables aléatoires tel que X suit la loi uniforme sur $[0, 2]$ et $\epsilon_i/X_i = W_i U_i + (1 - W_i)V_i$ avec W_i est une variable aléatoire binaire équiprobable ($P(W_i = 0) = P(W_i = 1) = 0.5$) et U_i est une variable aléatoire issue d'une loi $\mathcal{N}(X_i, 0.09)$ et V_i est une variable aléatoire qui suit $\mathcal{N}(0, 0.09)$. Dans ce deuxième modèle, la densité de la variable aléatoire Y sachant X est définie par:

$$f(y/x) = \frac{1}{0.6}\phi\left(\frac{y - 2\sin(\pi x)}{0.3}\right) + \frac{1}{0.6}\phi\left(\frac{y - 2\sin(\pi x) - x}{0.3}\right), \tag{11}$$

avec $\phi(.)$ est la densité d'une distribution normale centrée réduite.

Pour le reste des paramètres de l'application nous avons considéré ce qui suit:

- Le noyau K: $K \in \{$Gaussien, Epanechnikov $\}$.
- La discrétisation de y: y' varie entre -10 et 130 avec un pas $140/24$ (*ie* $J = 25$) dans le cas **Modèle 1**, et y' varie entre -2.5 et 2.5 avec un pas $5/24$ (*ie* $J = 25$) dans le cas **Modèle 2**.
- La taille des échantillon: $n \in \{50; 100; 150; 200; 250\}$.
- Le nombre d'échantillons: $N = 50$.

3.2. **Résultats numérique et graphique.**

Les résultats numériques obtenus dans le cadre du premier modèle sont rangés dans la table 1 et sont présentés dans les Figures 1 et 2. Tandis que les résultats numériques obtenus dans le cadre du deuxième modèle sont rangés dans la table 2 et sont présentés dans les Figures 3 et 4.

Table 1: Variation du AISE et du temps de calcul en fonction de n, cas du premier modèle.

K	$a \neq b$				$a = b$		
	n	(a^*, b^*)	ISE	$temps$ (mn)	h^*	ISE	$temps$ (mn)
Gaussien	50	(0.9351, 7.3807)	0.0028	19.2417	2.3412	0.0071	5.2667
	100	(0.8152, 6.5415)	0.0022	41.0333	1.9335	0.0055	9.7233
	150	(0.7866, 6.0262)	0.0018	60.5250	1.7712	0.0045	16.8017
	200	(0.7490, 5.6001)	0.0015	79.3617	1.6063	0.0037	25.1150
	250	(0.7151, 5.4174)	0.0013	102.5750	1.5425	0.0034	34.8233
Epanechnikov	50	(0.8877, 7.1378)	0.0029	2.3678	2.1058	0.0077	0.7190
	100	(0.8024, 6.1525)	0.0019	5.4912	1.7854	0.0055	1.7917
	150	(0.7744, 5.6500)	0.0018	8.0060	1.5944	0.0045	4.5634
	200	(0.7299, 5.3290)	0.0015	14.5177	1.5447	0.0040	6.0449
	250	(0.7054, 5.0261)	0.0013	18.5651	1.4397	0.0034	7.2891

Table 2: Variation du AISE et du temps de calcul en fonction de n, cas du deuxième modèle.

K	$a \neq b$				$a = b$		
	n	(a^*, b^*)	ISE	temps(mn)	h^*	ISE	temps(mn)
Gaussien	50	(0.0669, 0.3714)	0.1432	12.0698	0.1847	0.2152	5.1789
	100	(0.0540, 0.3109)	0.1114	22.1744	0.1443	0.1804	13.8123
	150	(0.0478, 0.2809)	0.0953	40.6394	0.1229	0.1623	19.1573
	200	(0.0439, 0.2576)	0.0816	60.7817	0.1081	0.1433	34.0199
	250	(0.0408, 0.2471)	0.0723	82.7658	0.1013	0.1338	38.7507
Epanechnikov	50	(0.0631, 0.3339)	0.1426	2.3489	0.1733	0.2295	1.3241
	100	(0.0504, 0.2930)	0.1101	10.1727	0.1360	0.1926	4.0519
	150	(0.0442, 0.2658)	0.0934	20.6833	0.1136	0.1708	7.7735
	200	(0.0408, 0.2420)	0.0798	28.4868	0.1006	0.1529	14.6891
	250	(0.0385, 0.2334)	0.0705	39.6346	0.0904	0.1407	18.5206

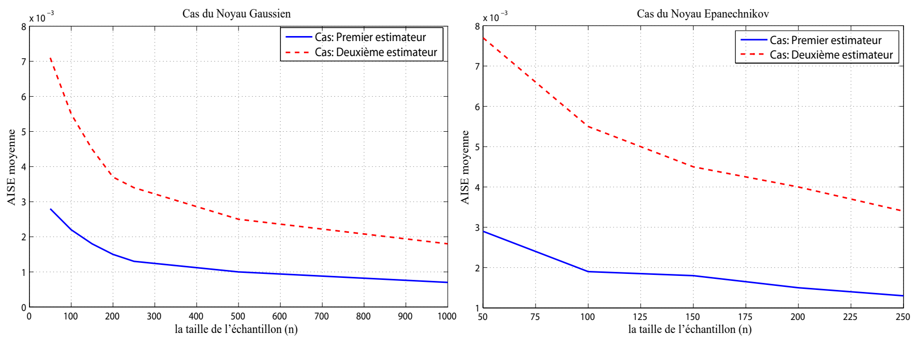

FIGURE 1. Variation du $AISE$ moyen en fonction de n, cas du premier modèle.

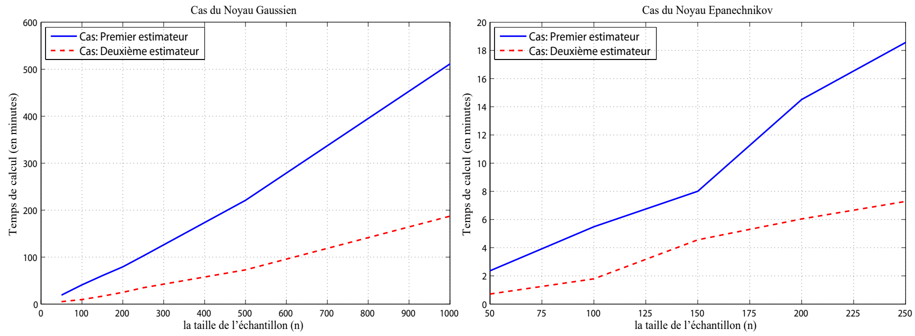

FIGURE 2. Variation du temps de calcul en fonction de n, cas du premier modèle.

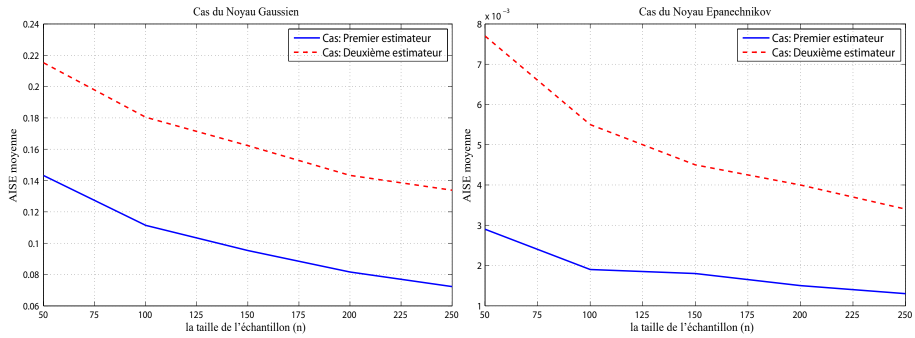

FIGURE 3. Variation du $AISE$ moyen en fonction de n, cas du deuxième modèle.

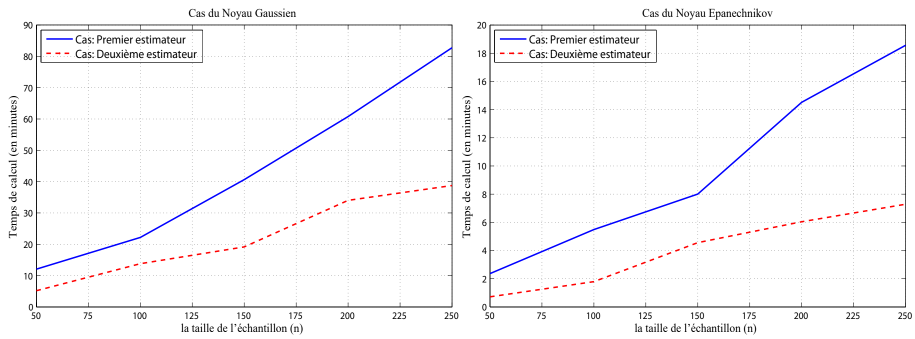

FIGURE 4. Variation du temps de calcul en fonction de n, cas du deuxième modèle.

3.3. **Discussion des résultats.** En tenant compte des résultats numériques et graphiques précédents on constate que:

- Les deux estimateurs convergent en fonction de la taille de l'échantillon n.

- Les estimateurs les plus performants au sens du AISE, dans le cas des deux modèles, sont obtenus lorsque nous considérons l'estimateur défini dans (1) et ceci quelque que soit la taille de l'échantillon et le noyau utilisé pour la construction de cet estimateur.
- La qualité des estimations au sens du critère retenus selon le noyau utilisé pour la construction de l'estimateur dépend du modèle et de l'estimateur considérés. En effet, dans le cas du premier modèle le choix du noyau pour la construction de l'estimateur (1) ou de l'estimateur (2) parait qu'il n'est pas d'une grande importance le fait que les deux noyaux (Normal et Epanechnikov) nous fournis des estimateurs pratiquement ayant le même AISE. Par contre, dans le deuxième modèle, si nous considérons l'estimateur défini dans (1) il est préférable de le construire via le noyau Epanechnikov et si nous considérons l'estimateur défini dans (2) alors dans ce cas il est préférable d'utiliser le noyau Normal.
- Le temps de calcul est plus considérable dans le cas du premier estimateur que le deuxième.
- Dans certains cas, l'investissement d'un temps de calcul pour améliorer la qualité de l'estimation n'est pas intéressant, car la contribution d'un temps supplémentaire dans la qualité de l'estimateur est très minime. En effet, par exemple, lorsque la taille de l'échantillon $n = 1000$ on constate clairement sur la figure 1 (à gauche) et la figures 2 (à gauche) que le gain d'une précision d'ordre 10^{-3} au sens de l'AISE pour le premier estimateur par rapport au deuxième estimateur, nécessite un temps de calcul supplémentaire dépassant cinq heures, ce qui est très fastidieux.
- Le temps de calcul lors de l'utilisation du noyau normal est plus considérable que dans le cas d'utilisation du noyau d'Epanechnikov ceci se justifier par la longueur de leurs support. Mais, les AISE moyennes engendrés par ces deux noyaux sont pratiquement les mêmes.

4. Conclusion

Dans ce travail à travers d'une application numérique, basée sur des échantillons simulés, nous avons mis en relief l'effet du choix de paramètre de lissage dans l'estimation à noyau d'une densité conditionnelle sous l'hypothèse d'égalité des deux paramètres de lissage de la direction de x et de la direction y ($a = b$) et le cas contraire ($a \neq b$).

Les résultats numériques et graphiques obtenus dans cette étude indiquent que les estimateurs les moins performants au sens du ISE moyenne sont obtenus dans le cadre d'hypothèse d'égalité des paramètres de lissage dans la direction de X et la direction de Y. Mais cette hypothèse s'avère intéressante lorsque la taille de l'échantillon est relativement grande le fait qu'un nous fournit dans un temps de calcul raisonnable des estimateurs pratiquement de même performances (au sens de l'ISE) que ceux obtenus lorsque cette hypothèse est niée.

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STABILITY OF CONTROLLED STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY G-BROWNIAN MOTION

MERIYAM DASSA* AND **ADEL CHALA***

ABSTRACT. In our proposed presentation, we will study the stability of controlled stochastic differential equations driven by G-Brownian motion (G-SDEs in short) with respect to the control variable by using the convex perturbation method, in which the set of admissible controls is convex. We aim to introduce three estimation's lemmas about the solution of controlled SDE within the framework of G-expectation. Lastly, we give the global form of the variational inequality, which is the principal tool to establish the G-stochastic maximum principle.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 93E20, 60G46, 93E10, 49K45, 49K35.

KEYWORDS AND PHRASES. Sublinear expectation, G-Brownian motion, G-expectation, G-stochastic differential equation, Variational inequality.

1. THE PROBLEM

How to measure uncertain quantities is an important problem. In 1953, when the Allais's paradox was introduced, the economists discovered that the theory of "expected utility" based on linear mathematical expectation was posed many questions. A question then arises: can we find a new theory that can be a natural generalization of a linear expectation? In particular, preserving, as much as possible, the properties of the classical linear expectation. As an answer to this question, Peng proposed a new notion of nonlinear expectation which is more dynamic, called sublinear expectation. As a typical case, Peng introduced G-expectation and a new type of Brownian motion called G-Brownian motion. After that the corresponding stochastic calculus of Itô's type was established. The existence and uniqueness of the solution of SDE driven by G-Brownian motion can be proved in a way parallel to that in the classical theory. But the stochastic optimal control within the framework of G-expectation becomes a challenging and fascinating problem.

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TESTS OF INDEPENDENCE AND GOODNESS-OF-FIT FOR COPULA MODELS WITH BIVARIATE CENSORED DATA

MOHAMED BOUKELOUA

Abstract. In a semiparametric copula model, we assume that the copula C of the studied distribution belongs to a parametric family $\{C_\theta, \theta \in \Theta\}$ ($\Theta \subset \mathbb{R}^d$) and that the margins are completely unknown. In this context, [3] proposed tests of independence of the margins based on the theory of divergences. The advantage of this approach is the fact that it works whatever the value of θ corresponding to the marginal independence is an interior or a boundary point of Θ. In the present work, we extend this approach to the case of bivariate censored data. So, we construct tests of independence and we establish the asymptotic distributions of the statistics of these tests under the null and the alternative hypotheses. We also propose Cramér-Von-Mises type goodness-of-fit tests for the parametric copula families and we study the asymptotic behavior of the statistics of these tests under the null and the alternative hypotheses.

2010 Mathematics Subject Classification. 62H15, 62N01, 62N03.

Keywords and phrases. Copulas, Tests of independence, Goodness-of-fit tests, Bivariate censored data.

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1. Empirical copula for censored data

Let $X = (X_1, X_2)$ be a couple of positive real random variables (r.r.v.), with a joint distribution function F and continuous margins F_1 and F_2, and let $R = (R_1, R_2)$ be a couple of positive censoring r.r.v. independent of X. The available observation consists in a sample $(Z_{1i}, Z_{2i}, \delta_{1i}, \delta_{2i})_{1\leq i\leq n}$ of independent copies of the vector $(Z_1, Z_2, \delta_1, \delta_2)$, where $Z_j = \min(X_j, R_j)$ and $\delta_j = 1_{\{X_j \leq R_j\}}$, $j \in \{1, 2\}$ ($1_{\{.\}}$ denotes the indicator function).

From now on, for any random variable V, F_V, S_V and T_V denote, respectively, the distribution function, the survival function and the upper endpoint of the support of V. Furthermore, for any right continuous function $H : \mathbb{R} \to \mathbb{R}$, we set $H(x^-) = \lim_{\varepsilon \overset{>}{\to} 0} H(x - \varepsilon)$ the left-hand limit of H at x when it exists. We assume that the copula C of X is twice continuously differentiable on $[0, 1]^2$, and that $T_{X_j} \leq T_{R_j}$ for $j \in \{1, 2\}$ which ensures that the variables X_1 and X_2 can be observed on the whole of their supports.

In the present work, we study the following bivariate censoring models.

Model I:

In this model, we assume that S_R can be written as $S_R(t_1, t_2) = C_R(S_{R_1}(t_1), S_{R_2}(t_2))$, where C_R is a known survival copula.[1] This model was studied by [6]. In this case, the empirical distribution function

$$\widetilde{F}_n(t_1,t_2)=\frac{1}{n}\sum_{i=1}^{n}1_{\{X_{1i}\leq t_1,X_{2i}\leq t_2\}}$$

can not be used to estimate $F(t_1,t_2)$ since X_1 and X_2 are not observed. Remarking that for any $t_1,t_2\in\mathbb{R}$

$$E\left(\delta_1\delta_2 g(Z_1,Z_2)1_{\{Z_1\leq t_1,Z_2\leq t_2\}}\right)=E\left(1_{\{X_1\leq t_1,X_2\leq t_2\}}\right)=F(t_1,t_2),$$

where $g(z_1,z_2)=P(R_1\geq z_1,R_2\geq z_2)^{-1}$, [6] proposed to replace $1_{\{X_{1i}\leq t_1,X_{2i}\leq t_2\}}$ by the observed quantity

$$\frac{\delta_{1i}\delta_{2i}}{C_R\left(\widehat{S}_{R_1}(Z_{1i}),\widehat{S}_{R_2}(Z_{2i})\right)}1_{\{Z_{1i}\leq t_1,Z_{2i}\leq t_2\}},$$

where

$$\widehat{S}_{R_1}(t)=\prod_{k/Z'_{1k}\leq t}\left(1-\frac{\sum_{j=1}^{n}1_{\{Z_{1j}=Z'_{1k},\delta_{1j}=0\}}}{\sum_{j=1}^{n}1_{\{Z_{1j}\geq Z'_{1k}\}}}\right)$$

($(Z'_{1k})_{1\leq k\leq m}$, $(m\leq n)$ being the distinct values of $(Z_{1i})_{1\leq i\leq n}$) is the Kaplan-Meier estimate of S_{R_1}, and $\widehat{S}_{R_2}$ is the Kaplan-Meier estimate of S_{R_2}, defined in the same way. This leads to the following estimate of $F(t_1,t_2)$.

$$F_n^{(I)}(t_1,t_2)=\frac{1}{n}\sum_{i=1}^{n}\frac{\delta_{1i}\delta_{2i}}{C_R\left(\widehat{S}_{R_1}(Z_{1i}),\widehat{S}_{R_2}(Z_{2i})\right)}1_{\{Z_{1i}\leq t_1,Z_{2i}\leq t_2\}}.$$

Denote by $\mathcal{X}_1$ (resp. $\mathcal{X}_2$) the support of X_1 (resp. X_2) and denote by $l^\infty(A)$ the space of all bounded real-valued functions defined on the nonempty set A. Applying Theorem 3.4. of [6] to the class of functions $\mathcal{F}=\{(t_1,t_2)\mapsto 1_{[0,x_1]\times[0,x_2]}(t_1,t_2),x_1\in\mathcal{X}_1,x_2\in\mathcal{X}_2\}$, we deduce that the process $\sqrt{n}(F_n^{(I)}-F)$ converges weakly in $l^\infty(\mathcal{X}_1\times\mathcal{X}_2)$, to a centered Gaussian process, under the following assumption.

Assumption I. We assume that

I.1. The first and the second partial derivatives of C_R are bounded on $[0,1]^2$. Moreover, $C_R(u_1,u_2)\neq 0$ for $u_1\neq 0$ and $u_2\neq 0$.

I.2. There exist $\alpha_1,\alpha_2\in[0,1]$ such that $C_R(u_1,u_2)\geq u_1^{\alpha_1}u_2^{\alpha_2}$.

I.3.

$$\int\frac{dF(t_1,t_2)}{C_R(S_{R_1}(t_1),S_{R_2}(t_2))}<\infty$$

and for some $a>0$ arbitrary small

$$\int\left[\frac{S_{R_1}^{1-\alpha_1}(t_1)\mathcal{K}_1^{1/2+a}(t_1)}{S_{R_2}^{\alpha_2}(t_2)}+\frac{S_{R_2}^{1-\alpha_2}(t_2)\mathcal{K}_2^{1/2+a}(t_2)}{S_{R_1}^{\alpha_1}(t_1)}\right]dF(t_1,t_2)<\infty,$$

where

$$\mathcal{K}_i(t)=\int_0^t\frac{dF_{R_i}(u)}{S_{R_i}(u)^2S_{X_i}(u)},i\in\{1,2\}.$$

[1]The survival copula C_R of R is defined by $C_R(u_1,u_2)=u_1+u_2-1+C^*(1-u_1,1-u_2)$, where C^* is the copula function of R.

Model II:
In this model, we assume that only X_1 is censored, in other words $R_2 = \infty$ almost surely (a.s.). This situation was studied by [7] who proposed the following estimate of F.

$$F_n^{(II)}(t_1, t_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{\delta_{1i}}{\widehat{S}_{R_1}(Z_{1i}^-)} 1_{\{Z_{1i}\le t_1, Z_{2i}\le t_2\}}.$$

This model is a particular case of model I for $C_R(u_1, u_2) = u_1u_2$ (the independence copula). So, the weak convergence of $F_n^{(II)}$ follows from Theorem 3.4. of [6] under the following assumption.
Assumption II. We assume that

$$\int \frac{dF(t_1, t_2)}{S_{R_1}(t_1^-)} < \infty$$

and for some $a > 0$ arbitrary small

$$\int \left[\int_0^{t_1} \frac{dF_{R_1}(u)}{S_{R_1}(u^-)^2 S_{X_1}(u)}\right]^{1/2+a} dF(t_1, t_2) < \infty.$$

Notice that assumptions **I.1** and **I.2** hold for the independence copula, taking $\alpha_1 = \alpha_2 = 1$.
Model III:
In this model, we assume that the difference between the censoring variables is observed, i.e., $R_2 = R_1 + \varepsilon$, where ε is an observed r.r.v., independent of R_1. This model was studied in [5]. Remarking that R_1 is right censored by $\max(X_1, X_2 - \varepsilon)$, and that the censorship indicator is $\eta = 1 - \delta_1\delta_2$, [5] used the same idea of [6], presented in model I, to propose the following estimate of F.

$$F_n^{(III)}(t_1, t_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{\delta_{1i}\delta_{2i}}{\widetilde{S}_{R_1}(\max(Z_{1i}, Z_{2i} - \varepsilon_i)^-)} 1_{\{Z_{1i}\le t_1, Z_{2i}\le t_2\}},$$

where $\widetilde{S}_{R_1}$ is the Kaplan-Meier estimate of S_{R_1} constructed from the sample $(\min(R_{1i}, \max(X_{1i}, X_{2i} - \varepsilon_i)), \eta_i)_{1\le i\le n}$.
Under the assumption
Assumption III. We have $E\left[S_{R_1}(\max(X_1, X_2 - \varepsilon)^-)^{-1}\right] < \infty$ and for some $a > 0$ arbitrary small, $E[\mathcal{C}^{1/2+a}(\max(X_1, X_2 - \varepsilon)^-) S_{R_1}(\max(X_1, X_2 - \varepsilon)^-)^{-1}] < \infty$, where

$$\mathcal{C}(t) = \int_0^t \frac{dF_{R_1}(u)}{S_{R_1}(u^-)^2 S_{X_1}(u)},$$

Theorem 3.1. of [5], applied to $\mathcal{F}$, entails that the process $\sqrt{n}(F_n^{(III)} - F)$ converges weakly in $l^\infty(\mathcal{X}_1 \times \mathcal{X}_2)$ to a centered Gaussian process. Notice that under assumption III, the class $\mathcal{F}$ satisfies assumption 2 of [5].

By analogy with the case of complete data, [4] proposed to estimate C by

$$\mathbb{C}_n^{(j)}(u_1, u_2) = F_n^{(j)}\left(\left(F_{1n}^{(j)}\right)^{-1}(u_1), \left(F_{2n}^{(j)}\right)^{-1}(u_2)\right), (u_1, u_2) \in [0,1]^2,$$

where $F_{1n}^{(j)}(t_1) = \lim_{t_2\to\infty} F_n^{(j)}(t_1, t_2)$ and $F_{2n}^{(j)}(t_2) = \lim_{t_1\to\infty} F_n^{(j)}(t_1, t_2)$, for $j \in \{I, II, III\}$ depending on the considered model.

Thanks to Theorem 2 of [4], we have for model (j) and under assumption (j), the censored empirical copula process $\sqrt{n}(\mathbb{C}_n^{(j)} - C)$ converges weakly, in $l^\infty([0,1]^2)$, to a tight, centered Gaussian process $G^{(j)}$. Otherwise, $\mathbb{C}_n^{(j)}$ being a left continuous function, it would be better to use the following right continuous empirical copula.

$$C_n^{(I)}(u_1,u_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{\delta_{1i}\delta_{2i}}{C_R\left(\widehat{S}_{R_1}(Z_{1i}), \widehat{S}_{R_2}(Z_{2i})\right)} 1_{\{F_{1n}^{(I)}(Z_{1i})\le u_1, F_{2n}^{(I)}(Z_{2i})\le u_2\}}, (u_1,u_2)\in[0,1]^2,$$

for model I. In the case of models II and III, the empirical copulas $C_n^{(II)}$ and $C_n^{(III)}$ can be defined in the same way, using the appropriate weights. Remark that for $j \in \{I, II, III\}$

$$\sup_{(u_1,u_2)\in[0,1]^2} \left| C_n^{(j)}(u_1,u_2) - \mathbb{C}_n^{(j)}(u_1,u_2)\right| = O_P\left(\frac{1}{n}\right)$$

(see [4]).
So, the process $\sqrt{n}(C_n^{(j)} - C)$ converges weakly, in $l^\infty([0,1]^2)$, to the limiting process $G^{(j)}$.

2. Semiparametric copula models

In a semiparametric copula model, we assume that the copula $C(u_1,u_2)$ has a parametric form $C_\theta(u_1,u_2)$, $\theta \in \Theta \subset \mathbb{R}^d$ and that the margins F_1 and F_2 are completely unknown. Let $c_\theta(u_1,u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C_\theta(u_1,u_2)$ be the density of $C_\theta(u_1,u_2)$ with respect to the Lebesgue measure on $\mathbb{R}^2$, and let θ_T and θ_0 denote respectively the true value of the parameter θ and the value that corresponds to the independence of the marginals, i.e., the value that satisfies $C_{\theta_0}(u_1,u_2) = u_1u_2$ (when it exists). To estimate θ_T, we use the theory of divergences and duality which we recall in the sequel. Let φ be a strictly convex, twice differentiable function defined from $\mathbb{R}$ to $[0,+\infty]$ such that its domain $\mathrm{dom}_\varphi = \{x \in \mathbb{R} \text{ such that } \varphi(x) < \infty\}$ is an interval with endpoints $a_\varphi < 1 < b_\varphi$ (which may be bounded or not, open or not). We assume that $\varphi(1) = 0$ and that φ is closed (i.e., if a_φ or b_φ is finite, then $\varphi(x) \to \varphi(a_\varphi)$, when $x \downarrow a_\varphi$, and $\varphi(x) \to \varphi(b_\varphi)$, when $x \uparrow b_\varphi$). For any probability measures P and Q defined on a measurable space $(E, \mathcal{B})$, the φ-divergence between Q and P, when Q is absolutely continuous with respect to P, is given by

$$D_\varphi(Q,P) = \int_E \varphi\left(\frac{dQ}{dP}(x)\right) dP(x).$$

Examples of divergence functions φ can be found in [1] and [3].

Applying a dual representation result of [1], [3] showed that the φ-divergence between C_{θ_0} and C_{θ_T} can be written as follows.

$$\begin{aligned} D_\varphi(\theta_0,\theta_T) &= \int_I \varphi\left(\frac{c_{\theta_0}(u)}{c_{\theta_T}(u)}\right) dC_{\theta_T}(u) \\ &= \sup_{\theta\in\Theta}\left\{\int_I \varphi'\left(\frac{1}{c_\theta(u)}\right) du_1du_2 - \int_I \left[\frac{1}{c_\theta(u)}\varphi'\left(\frac{1}{c_\theta(u)}\right) - \varphi\left(\frac{1}{c_\theta(u)}\right)\right] dC_{\theta_T}(u)\right\}, \end{aligned}$$

whenever

$$\int_I \left|\varphi'\left(\frac{1}{c_\theta(u)}\right)\right| du_1 du_2 < \infty \quad \text{for all } \theta \in \Theta,$$

where $I = (0,1)^2$ and $u = (u_1, u_2)$. Moreover, the sup is unique and is reached at $\theta = \theta_T$.
So, $D_\varphi(\theta_0, \theta_T)$ and θ_T can be estimated by analogy with the case of complete data ([3]), by the following plug-in estimates.

$$\widehat{D}_\varphi^{(j)}(\theta_0, \theta_T) = \sup_{\theta\in\Theta} \int_I m(\theta, u)\, dC_n^{(j)}(u)$$

and

$$\widehat{\theta}_\varphi^{(j)} = \arg\sup_{\theta\in\Theta} \left\{ \int_I m(\theta, u)\, dC_n^{(j)}(u) \right\},$$

where

$$\begin{aligned} m(\theta, u) &= \int_I \varphi'\left(\frac{1}{c_\theta(u)}\right) du_1 du_2 - \left\{ \frac{1}{c_\theta(u)} \varphi'\left(\frac{1}{c_\theta(u)}\right) - \varphi\left(\frac{1}{c_\theta(u)}\right) \right\} \\ &=: \int_I K_1(\theta, u) du_1 du_2 - K_2(\theta, u), \end{aligned}$$

for $j \in \{I, II, III\}$ depending on the considered bivariate censoring model.

If θ_T does not belong to the interior of Θ, this estimate may not be asymptotically normal. Therefore, it is not easy to test the independence hypothesis $\mathcal{H}_0 : \theta_T = \theta_0$ when θ_0 is a boundary point of Θ. To remedy this situation, we enlarge the parameter space Θ, as in [3], into a wider space $\Theta_e \supset \Theta$, so that θ_0 lies in the interior of Θ_e. The space Θ_e is defined by

$$\Theta_e = \left\{ \theta \in \mathbb{R}^d \text{ such that } \int_I \left|\varphi'\left(\frac{1}{c_\theta(u)}\right)\right| du_1 du_2 < \infty \right\},$$

and we redefine $\widehat{D}_\varphi^{(j)}(\theta_0, \theta_T)$ and $\widehat{\theta}_\varphi^{(j)}$ as follows.

$$\widehat{D}_\varphi^{(j)}(\theta_0, \theta_T) = \sup_{\theta\in\Theta_e} \int_I m(\theta, u)\, dC_n^{(j)}(u)$$

and

$$\widehat{\theta}_\varphi^{(j)} = \arg\sup_{\theta\in\Theta_e} \left\{ \int_I m(\theta, u)\, dC_n^{(j)}(u) \right\}.$$

3. Tests of independence

To test the hypothesis of marginal independence $\mathcal{H}_0 : \theta_T = \theta_0$ against the alternative $\mathcal{H}_1 : \theta_T \neq \theta_0$, we propose, by analogy with the case of complete data ([3]), the following test statistics.

$$T_{\varphi,n}^{(j)} = \frac{2n}{\varphi''(1)} \widehat{D}_\varphi^{(j)}(\theta_0, \theta_T), \quad j \in \{I, II, III\}.$$

We will establish the asymptotic distributions of $T_{\varphi,n}^{(j)}$ under $\mathcal{H}_0$ as well as under $\mathcal{H}_1$. For that, we need the following assumptions.
Denote by $\frac{\partial m}{\partial\theta}(., u)$ and $\frac{\partial^2 m}{\partial\theta^2}(., u)$, the gradient and the Hessian matrix of $m(., u)$, respectively.

H1: The functions $u \in I \mapsto \frac{\partial m}{\partial\theta}(\theta_T, u)$ and $u \in I \mapsto \frac{\partial^2 m}{\partial\theta^2}(\theta_T, u)$ are continuous from above and with discontinuities of the first kind.

H2: There exists a neighborhood $N \subset \Theta_e$ of θ_T, such that the first and the second partial derivatives with respect to θ of $K_1(\theta, u)$ are dominated on N by some integrable functions with respect to the Lebesgue measure on $\mathbb{R}^2$.

H3: The matrix $S = -\int_I (\partial^2/\partial\theta^2) m(\theta_T, u)\, dC_{\theta_T}(u)$ is non singular.

H4: The function $u \in I \mapsto m(\theta_T, u)$ is continuous from above and with discontinuities of the first kind.

Theorem 1. *For model (j), suppose that assumptions (j) and* **H1–H3** *are satisfied.*

i) *Under $\mathcal{H}_0$, the statistic $T_{\varphi,n}^{(j)}$ converges in distribution to $Y^\top Y$, where Y is a centered Gaussian vector, with covariance matrix $(\Sigma^{1/2})^\top M^{(j)} \Sigma^{1/2}$, where $\Sigma = S^{-1}$ and $\Sigma = \Sigma^{1/2} \left(\Sigma^{1/2}\right)^\top$ is the Cholesky decomposition of Σ.*

ii) *Under $\mathcal{H}_1$ and if assumption* **H4** *holds, then*

$$\sqrt{n}\left(\widehat{D}_\varphi^{(j)}(\theta_0, \theta_T) - D_\varphi(\theta_0, \theta_T)\right) \xrightarrow{\mathcal{D}} \int_I m(\theta_T, u)\, dG^{(j)}(u),$$

which is a centered Gaussian random variable.

In view of part i), the critical region of the test of independence, at level $\alpha \in (0,1)$, is $CR = \{T_{\varphi,n}^{(j)} > q_{1-\alpha}\}$, where $q_{1-\alpha}$ is the $(1-\alpha)$-quantile of the limiting distribution of $T_{\varphi,n}^{(j)}$. Part ii) of allows to approximate the power function $\theta_T \in \Theta \mapsto \pi(\theta_T) = P_{\theta_T}(CR)$. As in [3], we have

$$\pi(\theta_T) \approx 1 - F_{\mathcal{N}}\left(\frac{\sqrt{n}}{\sigma}\left(\frac{q_{1-\alpha}}{2n}\varphi''(1) - D_\varphi(\theta_0, \theta_T)\right)\right),$$

where σ^2 is the variance of $\int_I m(\theta_T, u)\, dG^{(j)}(u)$ and $F_{\mathcal{N}}$ is the cumulative distribution function of the standard normal distribution. Moreover, the sample size that ensures a desired power π is $\lfloor n_0 \rfloor + 1$, where

$$n_0 = \frac{a + b - \sqrt{a(a+2b)}}{2D_\varphi(\theta_0, \theta_T)^2},$$

with $a = \sigma^2 \left[F_{\mathcal{N}}^{-1}(1-\pi)\right]^2$ and $b = q_{1-\alpha}\varphi''(1) D_\varphi(\theta_0, \theta_T)$.

4. Goodness-of-fit tests

In this section, we are interested in test of fit of the hypothesis $\mathcal{H}_0 : C \in \{C_\theta, \theta \in \Theta\}$ against the alternative $\mathcal{H}_1 : C \notin \{C_\theta, \theta \in \Theta\}$. On the basis of the work of [4], we propose for model (j), the following Cramér-Von-Mises type statistics of test.

$$\Gamma_{\varphi,n}^{(j)} = n \int_I \left(C_n^{(j)}(u) - C_{\widehat{\theta}_\varphi^{(j)}}(u)\right)^2 dC_n^{(j)}(u), \quad j \in \{I, II, III\}.$$

In the next theorem, we give the asymptotic distribution of $\Gamma_{\varphi,n}^{(j)}$ under $\mathcal{H}_0$.

Theorem 2. *For model (j), we have under $\mathcal{H}_0$ and assumptions (j) and* **H1–H3**

i) *The process $\sqrt{n}\left(C_n^{(j)} - C_{\widehat{\theta}_\varphi^{(j)}}\right)$ converges weakly to a centered Gaussian process $\Lambda_\varphi^{(j)}$.*

ii) *The statistic* $\Gamma_{\varphi,n}^{(j)}$ *converges in distribution to* $\int_I \left(\Lambda_\varphi^{(j)}\right)^2(u)\,dC(u)$.

According to this theorem, the critical region of the test at level $\alpha \in (0,1)$ is $CR = \{\Gamma_{\varphi,n}^{(j)} > q_{1-\alpha}\}$, where $q_{1-\alpha}$ is the $(1-\alpha)$-quantile of the distribution of $\int_I \left(\Lambda_\varphi^{(j)}\right)^2(u)\,dC(u)$.

Now, we study the asymptotic behavior of $\Gamma_{\varphi,n}^{(j)}$ under $\mathcal{H}_1$. We need the following assumption.

H5 Assume that $C \notin \{C_\theta, \theta \in \Theta\}$ and that the pseudo-true value of θ

$$\theta_\varphi^* = \arg\sup_{\theta\in\Theta_e}\left\{\int_I m(\theta,u)\,dC(u)\right\}$$

exists and it is unique and satisfies

- The functions $u \in I \mapsto \frac{\partial m}{\partial\theta}(\theta_\varphi^*, u)$ and $u \in I \mapsto \frac{\partial^2 m}{\partial\theta^2}(\theta_\varphi^*, u)$ are continuous from above and with discontinuities of the first kind.
- There exists a neighborhood $N \subset \Theta_e$ of θ_φ^*, such that the first and the second partial derivatives with respect to θ of $K_1(\theta,u)$ are dominated on N by some integrable functions with respect to the Lebesgue measure on $\mathbb{R}^2$.
- The matrix $\int_I (\partial^2/\partial\theta^2) m(\theta_\varphi^*, u)\,dC_{\theta_\varphi^*}(u)$ is non singular.

Theorem 3. *For model (j), we have under assumptions (j) and* **H5**, $\Gamma_{\varphi,n}^{(j)}$ *tends in probability to infinity.*

This theorem shows that the Cramér-Von-Mises type test is consistent, i.e., its power tends to 1 as n tends to infinity.

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THE ALMOST COMPLETE CONVERGENCE OF THE CONDITIONAL HAZARD FUNCTION ESTIMATOR CASE ASSOCIATED DATA IN HIGH-DIMENSIONAL STATISTICS.

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ABSTRACT. we have in this paper, a study on the asymptotic properties of the kernel estimator of the conditional hazard function (introduced by Ferraty and Vieu (2000)) when the covariate is functional. The principal aim is the investigate of the convergence rate of the proposed estimator in case of functional quasi-associated data.

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1. INTRODUCTION

In recent decades, the statistical analysis of the functional data has attracted a lot of attention in the statistical mathematics. Such kind of data are used in a variety of fields including econometrics, epidemiology, environmental science and many others.
The monograph of Ferraty and Vieu (2006) is the first precursor in nonparametric functional statistics estimation. They focus in the estimation of the kernel method for conditional models and they established many asymptotic properties of regression, conditional quantile and conditional density estimator have been obtained. In this context, of functional nonparametric analysis, a lot of works are devoted to the estimations of the conditional hazard function in both: independent or dependent data.
The first results were obtained by Ferraty et al. (2003). They studied the almost complete convergence (with rate) of this model in several situations, including censored and/or dependent variables. For this topic, in the context of strong mixing dependence. Quintela-del-Río (2008) has shown that the kernel estimator presented by Ferraty et al. (2003) cited above is strongly consistent and asymptotically normally distributed. A generalization of these results in the spatial data case was obtained by Laksaci and Mechab (2010). More specifically, they studied the almost complete convergence of an adapted version of this estimator. The same authors have treated the L^2-convergence rate by giving the exact expression involved in the leading terms of the quadratic error and the asymptotic normality of the construct estimator (see, Laksaci and Mechab (2014)).
The quasi-association setting is a special case of weak dependence introduced by Doukhan and Louhichi (1999) for real-valued stochastic processes. It was applied by Bulinski and Suquet (2001) to real valued random fields and it

generalizes the positively associated variables introduced by Esary et al. (1967). The quasi-association dependency unifies both concepts (negative and positive association). Recall that, there are a lot of works dealing with the statistical analysis of positive and negative dependent random variables, we cite for example, Bulinski and Shabanovich (1998) and Newman (1984) and the references therein. Recently, there are few papers dealing with the nonparametric estimation for quasi-associated random variables. We quote, Douge (2010) studied a limit theorem for quasi-associated for random variables taking their values in a Hilbert space. Attaoui et al. (2015) studied the asymptotic results for an M-Estimator of the regression function for quasi-associated processes. Laksaci and Mechab (2016) studied the nonparametric relative regression for associated random variables. The main contribution of this work is the study of the estimator of the hazard function of Ferraty et al. (2008) in case of associated data. the almost-complete convergence [1] (a.co.) is established (with speed) of a kernel estimator for the hazard function of a real random variable conditioned by a functional explanatory variable. Note that, like all asymptotic statistics nonfunctional parametric, our result is related to the phenomenon of concentration of the probability measure of the explanatory variable and regularity of the functional space of the model. In this article, we discuss the asymptotic bias, dispersion of the estimator function of hazard in Quasi-Associated case. we recall the definition of Association:

definition 1. *A sequence $(X_n)_{n\in\mathbb{N}}$ of real random vectors variables is said to be Quasi-Association (QA), if for any disjoint subsets I and J of $\mathbb{N}$ and all bounded Lipschitz functions $f : \mathbb{R}^{|I|d} \to \mathbb{R}$ and $g : \mathbb{R}^{|J|d} \to \mathbb{R}$ satisfying*

$$Cov(f(X_i, i\in I), g(X_j, j\in J)) \le Lip(f)Lip(g)\sum_{i\in I}\sum_{j\in J}\sum_{k=1}^{d}\sum_{l=1}^{d}\left|Cov(X_i^k, X_j^l)\right|$$

where X_i^k denotes the k^{th} component of X_i,

$$Lip(f) = \sup_{x\neq y}\frac{|f(x)-f(y)|}{||x-y||_1} \text{ with } ||(x_1, ..., x_k)||_1 = |x_1| + \cdots + |x_k|.$$

definition 2. *Let $(\mathcal{H}, <.,.>)$ a separable Hilbert space with a orthonormal basis $e_k, k \ge 1$. A sequence $(X_n)n \in IN$ of real random variables taking values in $\mathcal{H}$ is said to be quasi-associated, with respect to the basis e_k if for any $d \ge 1$, the d-dimensional sequence $\{(<X_i, e_{j1}>, ..., <X_i, e_{jd}>), i \in \mathbb{N}\}$ is quasi-associated. Observe that the definition of quasi-association in the Hilbert space depends on the choice of the basis.*

The paper is organized as follows: the next section we present our model. Section 3 is dedicated to fixing notations and hypotheses. We state our main results in Section 4. The Section 5 is devoted the proofs of the auxiliary results.

[1]Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of real r.v.'s; we say that z_n converges almost completely (a.co.) to zero if, and only if, $\forall\epsilon > 0$, $\sum_{n=1}^{\infty}\mathbb{P}(|z_n| > \epsilon) < \infty$. Moreover, we say that the rate of almost complete convergence of z_n to zero is of order u_n (with $u_n \to 0$) and we write $z_n = O_{a.co.}(u_n)$ if, and only if, $\exists\epsilon > 0$, $\sum_{n=1}^{\infty} P(|z_n| > \epsilon u_n) < \infty$.

2. The model

Consider $Z_i = (X_i, Y_i)_{1\leq i\leq n}$ be a n quasi-associated random identically distributed as the random $Z = (X, Y)$, with values in $\mathcal{H} \times \mathbb{R}$, where $\mathcal{H}$ is a separable real Hilbert space with the norm $\| \, . \, \|$ generated by an inner product $< ., . >$.
We consider the semi-metric d defined by $\forall x, x' \in \mathcal{H}/d(x, x') = \| x - x' \|$.
In the following x will be a fixed point in $\mathcal{H}$ and $\mathcal{N}_x$ will denote a fixed neighborhood of x and $\mathcal{S}$ will be fixed compact subset of $\mathbb{R}$.
We intend to estimate the conditional hazard function h^x using n dependent observations $(Z_i)_{i\in\mathbb{N}}$ draw from a random variables with the same distribution with Z where the regular version F^x of the conditional distribution function of Y given $X = x$ exists for any $x \in N_x$. Moreover we suppose that F^x has a continuous density f^x with respect to (w.r.t) Lebesgue's measure over $\mathbb{R}$. we define the function hazard h^x, for $y \in \mathbb{R}$ and $F^x(y) < 1$, by

$$h^x(y) = \frac{f^x(y)}{1 - F^x(y)}, \tag{1}$$

To this aim, we first introduce the kernel type estimator $\widehat{F}^x$ of F^x defined by

$$\widehat{F}^x(y) = \frac{\sum_{i=1}^n K(h_K^{-1} d(x, X_i)) H(h_H^{-1}(y - Y_i))}{\sum_{i=1}^n K(h_K^{-1} d(x, X_i))}, \qquad \forall y \in \mathbb{R} \tag{2}$$

where K is the kernel, H is a given distribution function and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) is a sequence of positive real numbers.
We define the kernel estimator $\widehat{f}^x$ of f^x by:

$$\widehat{f}^x(y) = \frac{h_H^{-1} \sum_{i=1}^n K(h_K^{-1} d(x, X_i)) H'(h_H^{-1}(y - Y_i))}{\sum_{i=1}^n K(h_K^{-1} d(x, X_i))}. \tag{3}$$

Where H' is the derivative of H.
Finally, the estimator of the conditional hazard function is $\widehat{h}^x$ defined by

$$\widehat{h}^x(y) = \frac{\widehat{f}^x(y)}{1 - \widehat{F}^x(y)} . \forall y \in \mathbb{R}. \tag{4}$$

3. Notations and hypotheses

All along the paper, when no confusion will be possible, we will denote by C or/and C' some strictly positive generic constants whose values are allowed to change. The variable x is a fixed point in $\mathcal{H}$, $\mathcal{N}_x$ is a fixed neighborhood of x. We assume that the random pair $Z_i = \{(X_i, Y_i), i \in \mathbb{N}\}$ is stationary quasi-associated processes. Let λ_k the covariance coefficient defined as:

$$\lambda_k = sup_{s\geq k} \sum_{|i-j|\geq s} \lambda_{i,j}$$

where

$$\lambda_{i,j} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \mid cov(X_i^k, X_j^l) \mid + \sum_{k=1}^{\infty} \mid cov(X_i^k, Y_j) \mid + \sum_{l=1}^{\infty} \mid cov(Y_i, X_j^l) \mid + cov(Y_i, Y_j) \mid .$$

X_i^k denotes the k^{th} component of X_i defined as $X_i^k := < X_i, e^k >$.
For $h > 0$, let $B(x,h) := \{x' \in \mathcal{H}/d(x',x) < h\}$ be the ball of center x and radius h.
To establish the almost complete convergence of the estimator $\widehat{h}^x$ we need to include the following assumptions:

(**H1**) $\mathbb{P}(X \in B(x,h)) = \phi_x(h) > 0$ and the function $\phi_x(h)$ is a differentiable at 0.

(**H2**) The conditional density $f^x(y)$ satisfies the Hölder condition, that is : $\forall (x_1,x_2) \in \mathcal{N}_x \times \mathcal{N}_x$, $\forall (y_1,y_2) \in \mathcal{S}^2$

$$|F^{x_1}(y_1) - F^{x_2}(y_2)| \leq C\left(d^{b_1}(x_1,x_2) + |y_1 - y_2|^{b_2}\right), \quad b_1 > 0,\, b_2 > 0$$

and

$$|f^{x_1}(y_1) - f^{x_2}(y_2)| \leq C\left(d^{b_1}(x_1,x_2) + |y_1 - y_2|^{b_2}\right), \quad b_1 > 0,\, b_2 > 0.$$

where S is a fixed compact subset of $\mathbb{R}$.

(**H3**) The kernel H is a differentiable function and H' is a positive, bounded, Lipschitzian continuous function such that:

$$\int |t|^{b_2} H'(t)dt < \infty \quad \text{and} \int H'^2(t)dt < \infty.$$

(**H4**) K is a bounded continuous Lipschitz function such that:

$$C\mathbf{1}_{[0,1]}(.) < K(.) < C'\mathbf{1}_{[0,1]}(.)$$

where $\mathbf{1}_{[0,1]}$ is a indicator function.

(**H5**) The sequence of random pairs $(X_i, Y_i), i \in \mathbb{N}$ is quasi-associated with covariance coefficient λ_k, $k \in \mathbb{N}$ satisfying :

$$\exists \alpha > 0, \exists C > 0, \text{such that } \lambda_k \leq Ce^{-\alpha k}$$

(**H6**) for all pairs (i, j), the joint distribution functions

$$\Psi_{i,j}(h) = \mathbb{P}\left[(X_i, X_j) \in B(x,h) \times B(x,h)\right]$$

satisfy

$$0 < \sup_{i \neq j} \Psi_{i,j}(h) = O(\phi_x^2(h_k))$$

(**H7**) The bandwidths h_K and h_H are a sequences of positive numbers satisfying:

$$\lim_{n\to\infty} \frac{log^5 n}{nh_H^j \phi_x(h_K)} = 0, j = 0, 1.$$

4. Main result: Pointwise almost complete convergence

Theorem 4.1. *Under hypotheses (H1)-(H7), we have:*

$$|\widehat{h}^x(y) - h^x(y)| = O\left(h_K^{b_1} + h_K^{b_2}\right) + O_{a.co}\left(\left(\frac{logn}{nh_H\phi_x(h_K)}\right)^{\frac{1}{2}}\right). \tag{5}$$

Proof

The proof of theorem 4.1 is based on the following lemmas:

Lemma 4.2. *Under hypotheses (H1)-(H4) and (H6), we have:*

$$\frac{1}{\widehat{F}_D^x}(\widehat{F}_N^x(y) - \mathbb{E}\widehat{F}_N^x(y)) = O_{a.co}\left(\left(\frac{logn}{n\phi_x(h_K)}\right)^{\frac{1}{2}}\right). \tag{6}$$

Corollary 4.3. *Under hypotheses (H1)-(H4) and (H6), we have:*

$$\sum_{i=1}^{\infty}\mathbb{P}\left(\mid \widehat{F}_D^x \mid < 1/2\right) < \infty. \tag{7}$$

Lemma 4.4. *Under hypotheses (H1)-(H6), we have:*

$$\frac{1}{\widehat{F}_D^x}(F^x(y) - \mathbb{E}\widehat{F}_N^x(y)) = O\left(h_K^{b_1} + h_K^{b_2}\right). \tag{8}$$

Lemma 4.5. *Under hypotheses (H1)-(H3) and (H6) , we have:*

$$\widehat{F}_D^x(y) - \mathbb{E}(\widehat{F}_D^x(y) = O_{a.co}\left(\left(\frac{logn}{n\phi_x(h_K)}\right)^{\frac{1}{2}}\right) \tag{9}$$

Lemma 4.6. *Under hypotheses (H1)-(H6), we have:*

$$\frac{1}{\widehat{F}_D^x}(f^x(y) - \mathbb{E}\widehat{f}_N^x(y)) = O\left(h_K^{b_1} + h_K^{b_2}\right). \tag{10}$$

Lemma 4.7. *Under hypotheses (H1)-(H4) and (H6), we have:*

$$\frac{1}{\widehat{F}_D^x}(\widehat{f}_N^x(y) - \mathbb{E}\widehat{f}_N^x(y)) = O_{a.co}\left(\left(\frac{logn}{nh_H\phi_x(h_K)}\right)^{\frac{1}{2}}\right). \tag{11}$$

Lemma 4.8. *Under hypotheses of the Theorem 4.1, we have:*

$$\exists\delta > 0, \sum_{i=1}^{\infty}\mathbb{P}\left\{|1 - \widehat{F}^x(y)| < \delta\right\} < \infty. \tag{12}$$

5. Auxiliary results

First of all, we state the following lemmas.

Lemma 5.1. *(See, Douge (2010)) Let $(X_n)_{n\in\mathbb{N}}$ be a quasi-associated sequence of random variables with values in $\mathcal{H}$. Let $f \in BL(\mathcal{H}^{|I|}) \cap \mathbb{L}^\infty$ and $g \in BL(\mathcal{H}^{|J|}) \cap \mathbb{L}^\infty$,for some finite disjoint subsets $I, J \in \mathbb{N}$. Then*

$$Cov\left(f\left(X_i, i \in I\right), g(X_j, j \in J)\right) \leq Lip(f)Lip(g)\sum_{i\in I}\sum_{j\in J}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left|Cov(X_i^k, X_j^l)\right|$$

where $(BL(\mathcal{H}^u; u > 0)$ *is the set of bounded Lipschitz functions* $f : \mathcal{H}^u \to \mathbb{R}$ *and* $\mathbb{L}^\infty$ *is the set of bounded functions.*

Lemma 5.2. *(See, Kallabis and Newmann (2006)).*
Let $X_1, ..., X_n$ *the real random variables such that* $\mathbb{E}(X_j) = 0 and\, \mathbb{P}\left(| X_j |\leq M\right) = 1\; for\; all j = 1, ..., n and\; some M < \infty$, *Let* $\sigma_n^2 = Var\left(\sum_{i=1}^n \Delta_i\right)$.
Assume,furthermore, that there exist $K < \infty$ *and* $\beta > \infty$ *such that, for all u-uplets* $(s_1, ..., s_u) \in \mathbb{N}^u$, $(t_1, ..., t_v) \in \mathbb{N}^v$ *with* $1 \leq s_1 \leq ... \leq s_u \leq t_1 \leq ... \leq t_v \leq n$. *the following inequality is fulfilled :*

$$| cov(X_{s_1}...X_{s_u}, X_{t_1}...X_{t_v}) |\leq K^2 M^{u+v-2} v e^{-\beta(t_1-s_u)}.$$

. *Then,*

$$\mathbb{P}\left(| \sum_{j=1}^n X_j |> t\right) \leq exp\left\{-\frac{t^2/2}{A_n + B_n^{1/3} t^{5/2}}\right\}$$

for some :

$$A_n \leq \sigma_n^2$$

and

$$B_n = \left(\frac{16nK^2}{9An(1-e^-\beta) \vee 1}\right)\frac{2(K \vee M)}{1-e^-\beta}.$$

6. Conclusion

In this communication, we present, we established the consistency properties (with rates) of the conditional hasard function with a functional explicatory variable for quasi-associated data, the pointwise almost complete convergence (with rates) of the kernel estimate of this model are obtained, For further work it will be interesting to establish the consistency properties (with rates) of the conditional density function with a functional explicatory variable for quasi-associated and censored data.

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THE EXISTENCE RESULT OF SOLUTION FOR G-STOCHASTIC DIFFERENTIAL EQUATION

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Abstract. In this poster we preset the existence and the uniqueness of the solution of system of stochastic differential equations driven by G-Brownian motion by using the Caratheodory approximation scheme.

2010 Mathematics Subject Classification. 60H05, 60H10, 60H99.

Keywords and phrases.G-expectation, G-brownian motion, G-stochastic differential equations, Caratheodory approximation scheme.

1. Define the problem

The Caratheodory approximation scheme has been used by several mathematicians to prove the existence theorem for the solutions of ordinary differential equations under mild regularity conditions. N. Caratheodory [2] was the first to introduce this approximation for ordinary differential equations.

The existence and the uniqueness of the solution X_t, for $G-$SDEs (1.1) under different conditions was proved in ([1], [4], [5], [7] and [8]).

In this poster, we present the existence and the uniqueness of the solution for the following system of stochastic differential equations driven by a $G-$Brownian motion (S$G-$SDEs):

$$\begin{cases} X_t = X_0 + \int_0^t f_1(s, X_s, Y_s)\, ds + \\ + \int_0^t f_2(s, X_s, Y_s)\, d\langle B\rangle_s + \int_0^t f_3(s, X_s, Y_s)\, dB_s \\ Y_t = Y_0 + \int_0^t g_1(s, X_s, Y_s)\, ds + \\ + \int_0^t g_2(s, X_s, Y_s)\, d\langle B\rangle_s + \int_0^t g_3(s, X_s, Y_s)\, dB_s \end{cases}$$

Where (X_0, Y_0) is a given initial condition, $(\langle B_t\rangle)_{t\geq 0}$ is the quadratic variation process of the $G-$Brownian motion $(B_t)_{t\geq 0}$ and all the coefficients $f_i(t,x,y)$, $g_i(t,x,y)$, for $i = 1,2,3$, satisfy the Lipschitz and the linear growth conditions with respect to (x,y) where the constants are time dependant.

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The conditional tail expectation of a heavy-tailed distribution under random censoring

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Abstract The conditional tail expectation (CTE) has the advantage, over the very popular Value-at-Risk, of being a coherent risk measure. Hence, it has become a very useful tool in financial and actuarial risk assessment. For such quantity, [1] discussed the sample estimator and [2] proposed an estimator for an important class of Pareto-like distributions. In this paper, we consider data that are heavy-tailed and, at the same time, randomly censored. By making use of survival and extreme value methodologies, we define an estimator for the CTE and we construct confidence intervals and discuss their lengths and coverage probabilities. Finally, we apply our results to a set of real data, namely the survival times of Australian male Aids.

Keywords: Coherent risk measure; Conditional tail expectation; Extreme value index; Heavy-tails; Hill estimator; Kaplan-Meier estimator.

1 Simulation study

We carry out a simulation study to illustrate the performance of our estimator, through two sets of data from Burr (ξ, η) and Fréchet (ξ) models respectively defined, for $x \geq 0$, by

$$\bar{F}(x) = \left(1 + x^{1/\eta}\right)^{-\eta/\xi} \text{ and } \bar{F}(x) = 1 - \exp(-x^{-1/\xi}),$$

where ξ and η are two positive parameters.

The confidence intervals are constructed by the technique bootstrap, we use the percentile confidence intervals method.

2 Case study

In this section, we apply our estimation procedure to the dataset known as Australian Aids data and provided by Dr P.J. Solomon and the Australian National Centre in HIV Epidemiology and Clinical Research. It consists in medical observations on 2843 patients (among whom 2754 are male) diagnosed with Aids in Australia before July 1^{st}, 1991 (See [3] and [4]).

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THE PERFORMANCE EVALUATION OF THE PATTERN INFORMATICS METHOD: A RETROSPECTIVE ANALYSIS FOR JAPAN AND THE IBERO-MAGHREB REGIONS

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Abstract. Forecasting aims to estimate the probability of earthquakes occurrence in a given space-time volume. Among the typical examples of forecasting methods: the Pattern Informatics (PI) method which is based on the identification of the significant variations in space-time seismicity rate. In this study, Japan and the Ibero-Maghrebian earthquake catalog are downloaded from the United States Geological Survey (USGS) website. The Pattern Informatics method is applied to forecast large earthquakes of magnitude $m \geq m_t$, in which the completeness magnitude $m \geq m_c$. Results are presented as regional forecasting maps showing areas with small, moderate and high probabilities of future target earthquakes occurrence. The Relative Operating Characteristics (ROC) and Molchan diagrams are used to evaluate the performance of the Pattern Informatics method.

1. Define the problem

Define the problem Earthquakes are among the most natural disasters as they pose a major risk. Indeed, one major earthquake can lead to large human (dead, vagabonds, wounded) and material losses (building, houses, hospitals,...). In this context, to avoid the occurrence of earthquakes, we use the earthquake forecasting topic, as a step towards mitigation of losses from casualties. In practice, forecasting aims to estimate earthquake occurrence probability in a given space-time volume. Pattern Informatics method is among the forecasting methods that we rely on in our study. It has been developed by Rundle et al. (2002), Tiampo et al. (2002a, b, c), and Holliday et al. (2006). This method is used to identify the space-time seismicity rate variations that occurred in the past, and it can detect precursor seismic activation or quiescence. For the PI analysis, the study regions are divided into a grid of equal-size cells, with cell size l°* l°, the main input parameter to estimate the seismic hazard map of any region is an earthquake catalogue. In practice, the Ibero-Maghrebian and Japan regions are selected as study regions, which are earthquake-prone areas, the earthquake catalogues are downloaded from the United States Geological Survey (USGS) website, with completeness magnitude m mc. Furthermore, forecasting results are resumed as PI forecasting maps with hotspots covering regions where target earthquakes of magnitude m m_t are mostly expected to occur. Finally, the results testing plays a very important role in evaluating the PI method performance. Here, we apply the Relative Operating Characteristic (ROC) and Molchan diagrams.

2010 Mathematics Subject Classification. 42C05, 33C45.

Earthquakes Forecasting, Pattern Informatics, Forecast verification.

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THE PROPERTIES OF THE STOCHASTIC FLOW GENERATED BY THE ONE-DEFAULT MODEL IN MULTI-DIMENSIONAL CASE

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Abstract. In our research we will look at the differentiability of the solution of one-default model with respect to initial value in multi-dimensional case. Precisely, We show the existence of the partial derivative in the initial value basing on the idea of H-Kunita, R.M-Dudley and F-Ledrappier [2].

1. Introduction

We consider a following stochastic differential equation:

$$(\natural_u) = \begin{cases} dX^x_{u,t} = X^x_{u,t}\left(-\frac{e^{-\Lambda_t}}{1-Z_t}N_t + f(X_t - (1-Z_t))dY_t\right), \; t \in [u,\infty[, \\ X^x_{u,u} = x, \end{cases}$$

where x is the initial condition.
This equation is called $\natural-$equation which is the priceless system in financial mathematics and it's one of the best ways to represent the evolution of a financial market after the default time, it's considered a prosperous system of parameters (Z, Y, f). the parameter Z determines the default intensity. The parameters Y and f describe the evolution of the market after the default time τ.
Let's move to the multidimensional version of $\natural-$equation [3]. On a probability space $(\Omega, (\mathbb{F})_{t\geq 0}, \mathbb{P})$. We have:

$$(\natural_u) = \begin{cases} dX_{u,t}(x) = X_t(x)\left(-\frac{e^{-\Lambda_t}}{1-Z_t}dN_t + F(X_t(x) - (1-Z_t))dY_t\right), \; t \in [u,\infty[, \\ X_{u,u}(x) = x, \end{cases}$$

Where $(\Lambda^1, ..., \Lambda^d)$ is $d-$dimensional is continuous increasing process null at the origin, $N_t = (N^1, ..., N^d)$ is a given $d-$dimensional continuous non-negative local martingale such that $0 < Z_t = N_t e^{-\Lambda_t} < 1, t > 0$ and $(Z(t,w) = (Z^1(t,w), ..., Z^d(t,w))$ presents the default intensity. $(Y(t,w) = (Y^1(t,w), ..., Y^n(t,w))$ is a given $n-$dimensional continuous local martingale and $F = (F_1, ..., F_n)$ on $\mathbb{R}^n$ is Lipschitz mapping null at the origin.

Key words and phrases. Credit risk; Stochastic flow; Stochastic differential equations; Diffeomorphism.

This equation has a unique solution $X_{u,t}(x)$ such as;

$$X_t^u = x + \int_u^t X_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_s} \right) dN_s + \int_u^t X_s \sum_{i=1}^{d} \sum_{j=1}^{n} F^{ij}(X_s - (1 - Z_s))dY_s^j,\ s \in [u, t]$$

where $X_u^u = x$ is the initial condition and F^{ij} is $i - th$ component of the vector function F^j.

2. Properties Of Stochastic Flows.

Let $\zeta_{s,t}(x, \omega); s, t \in [0, T], x \in \mathbb{R}^d$ be continuous $\mathbb{R}^d-$valued random field defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Definition 2.1. *Stochastic flow of homeomorphisms(or simply a flow) is a map $\zeta_{s,t}(\omega) \equiv \zeta_{s,t}(., \omega)$ defines from $\mathbb{R}^d$ into itself for almost all ω indexed by tow parameters s and t such that $s < t$, the first represents the initial time of the flow and the second represents the state of the flow and it satisfies the following properties:*

(1) For any $x \in \mathbb{R}^d$, $\zeta_{s,t}(\omega)$ is continuous.
(2) The map $\zeta_{s,t} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a homeomorphism for any s, t.
(3) $\zeta_{s,t}(\omega)$ is $k-$times continuously differentiable with respect to x for all $s, t \in \mathbb{R}^d$.
(4) $\zeta_{t,u}(\zeta_{s,t}(\omega)) = \zeta_{s,u}(\omega)$ for any s, u, t and any x. and $\zeta_{s,s}(\omega) = Id_{\mathbb{R}^d}$ for any s.

If additionally $\zeta_{s,t}(\omega)$ satisfies also properties (2)and (3). It is called stochastic flow C^k-diffeomorphisms.

3. Description of the work to be carried out

In our work we had demonstrated the differentiability of the solution of the one-default model in multi-dimensional case under the following hypothesis: the coefficients of $\natural-$equation are Lipschitz and the processes represented in this equation take real values.

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THRESHOLD SPATIAL NON-DYNAMIC PANEL DATA

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Abstract. In this work, we introduce a new threshold spatial non-dynamic panel data model, which extends the classical spatial panel data (SPD) with fixed effects. We introduce a threshold variable to examine the non-linearity and the heterogeneity of the spatial effects in SPD models. We first provide a quasi-maximum likelihood (QML) method to estimate the parameters of the proposed model. We thus develop linearity test. This test occupy a prominent place and guides us in the choice of specification to take into account the non-linearity if it exists. We finally show, through a Monte Carlo study, that the proposed QML estimation and test procedures provide good results.

2010 Mathematics Subject Classification. 91B72, 62P20.

Keywords and phrases. Spatial model, Panel data,Threshold model, Linearity test.

1. Define the problem

Panel data models with spatial interactions have received a lot of interest since the work of Anselin (1988) in the field of econometrics. A number of different settings, such as model with spatial lag or spatial error, static or dynamic, fixed or random individual effect, temporal effect, have been explored and their corresponding estimation methods were established.
Indeed, Aquaro et al. (2015) have introduced regional heterogeneity for spatial panel data. Deng (2018) have proposed a threshold spatial autoregressive (TSAR) model with varying spatial parameters for different regimes. In this model, the slopes of all exogenous regressors remain the same for different regimes. Zho et al. (2020) generalized the TSAR model by considering a threshold spatial Durbin (TSD) model, which allows for heterogeneous slope coefficients for both spatial lags and all exogenous. It should be pointed out that TSAR and TSD models are very useful but limited to cross-section data modeling.
On the other hand, it is widely documented that many extensions and mathematical developments of threshold models have been adopted for the analysis of panel data structures. Hansen (1999) proposed a panel threshold regression (PTR) model for the non-dynamic panel case. His main contribution lies in the possibility of allowing the individuals constituting the panel to be in different regimes during a given period. This enables the heterogeneity in the panel to be better captured and allows for a visualization of the non-linearity in the interaction between the dependent variable and the explanatory variables for each panel's component. However, this study do not explore estimation and testing issues under spatial dependence. In this work, we extend the threshold model to a spatial panel with fixed effects,

where we applied to spatial panel data the same approach as is usually used in threshold time series models. In this model, an individual may have a dynamic different from the others. Through this threshold regime switching mechanism, we analyze the heterogeneity in the spatial panel and how the exogenous threshold variable impacts the dependent variable.
We consider a threshold spatial non-dynamic panel data with two regimes where each regime is represented by a classical balanced spatial panel data with spatial dependence.
The model is defined as:

$$
\begin{aligned}
y_{i,t} &= \begin{cases} \lambda_1 \sum\limits_{j=1}^{n} w_{i,j} y_{j,t} + x_{i,t}\beta_1 + \varepsilon_{i,t} & if \quad q_{i,t} \le \gamma \\ \lambda_2 \sum\limits_{j=1}^{n} w_{i,j} y_{j,t} + x_{i,t}\beta_2 + \varepsilon_{i,t} & if \quad q_{i,t} > \gamma \end{cases} \\
\varepsilon_{i,t} &= \mu_i + v_{i,t}
\end{aligned} \tag{1}
$$

Where the subscript i represent cross-section $(i = 1, 2, ..., n)$ and t is the time periods $(t = 1, 2, ..., T)$. $y_{i,t}$ is a scalar dependent variable, the variable $\sum\limits_{j=1}^{n} w_{i,j} y_{j,t}$ denote the interaction effect of the dependent variable $y_{i,t}$ with the dependent variable $y_{j,t}$ in neighbouring unit and $w_{i,j}$ is the $(i, j)\, th$ element of a constant spatial weight matrix W. λ_i $(i = 1, 2)$ scalars of these endogenous interaction, $x_{i,t}$ is a $(1 \times k)$ vector of exogenous variable, β_i $(i = 1, 2)$ represents the slope coefficients that differ for each regime, $v_{i,t}$ is a independent and identically distributed variable with 0 mean and σ^2 finite variance , μ_i is the individual effect. $q_{i,t}$ is the threshold variable and γ is the threshold parameter.

In this communication, we describe a straightforward procedure for estimating our threshold spatial non-dynamic panel model via a quasi-maximum likelihood method. We deal also with the issue of inference in our spatial panel data framework where we present the test of linearity based on a bootstrap approach. We finally show, through a Monte Carlo study, that the proposed QML estimation and test procedures provide good results.

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Résumé

Dans cette travaille, nous nous proposons une nouvelle distribution de durée de survie basé sur les modèles tronqués cette distribution est nommée la distribution de Poison Peudo Lindley tronquée à zero. Cette distribution dépend de deux paramètres, l'un est un paramètre de forme et l'autre c'est un paramètre d'échelle. Nous exposons aussi l'étude des estimateurs on utilise une approche classique du maximum de vraisemblance et la méthode des moments. Dans le cas de l'approche classique, les estimateurs sont des solutions d'un système non linéaire dont les solutions ne sont pas explicites analytiquement, des méthodes numériques sont été adoptées. Finalement, une étude par simulation et une analyse de données réelles ont été réalisées pour comparer le

modèle introduit avec d'autres modèles tronqués à un seul et à deux paramètres.

UNIFORM CONSISTENCY OF A NONPARAMETRIC RELATIVE ERROR REGRESSION ESTIMATOR FOR FUNCTIONAL REGRESSORS UNDER RIGHT CENSORING

OMAR FETITAH, IBRAHIM M. ALMANJAHIE, MOHAMMED KADI ATTOUCH, AND ALI RIGHI

ABSTRACT. In this paper, we investigate the asymptotic properties of a nonparametric estimator of the relative error regression given a functional explanatory variable, in the case of a scalar censored response, we use the mean squared relative error as a loss function to construct a nonparametric estimator of the regression operator of these functional censored data. We establish the strong almost complete convergence rate and asymptotic normality of these estimators. A simulation study is performed to illustrate and compare the higher predictive performances of our proposed method to those obtained with standard estimators.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 62G05, 62G08, 62G20, 62G35, 62N01.

KEYWORDS AND PHRASES. Relative error regression, Censored data, nonparametric kernel estimation, functional data analysis, almost complete convergence, asymptotic normality, small ball probability.

1. INTRODUCTION

Modeling functional variables have received increasing interest in the last few years from mathematical or application points of view. There are many results for nonparametric models for more details on the subject, and we refer the reader to the monograph of [5].

The study of a scalar response variable Y given a new value for the explanatory variable X is an important subject in nonparametric statistics. This regression relation is modeled by:

$$Y = r(X) + \epsilon, \tag{1}$$

where $r(\cdot)$ is the regression function and ϵ a sequence of error independent to X.

Usually, $r(\cdot) = \mathbb{E}[Y|X]$ is estimated by minimizing the mean squared loss function. However, this loss function is based on some restrictive conditions that is the variance of the residual is the same for all the observations, which is inadequate when the data contains some outliers.

When the predicted values are large or when the data contain many outliers, the following criterium

$$\mathbb{E}\left[\left(\frac{Y - r(X)}{Y}\right)^2 |X\right], for\, Y > 0 \tag{2}$$

1

is a more meaningful measure of the prediction performance than the least square error. Notice that this kind of model, so-called relative error regression, has been widely studied in parametric regression analysis. When the first two conditional inverse moments of Y given X are finite, the solution is given by the minimization of the sum of absolute relative errors for a linear model of the following ratio:

$$r(x) = \frac{\mathbb{E}[Y^{-1}|X=x]}{\mathbb{E}[Y^{-2}|X=x]}. \tag{3}$$

The least absolute relative error estimation for multiplicative regression models was proposed by [3], who proved consistency and asymptotic normality of their estimator and also provided an inference approach via random weighting. [9] discussed the asymptotic efficiency of relative logistic regression in a parametric context, particularly when explanatory variables are normally distributed. Moreover, [6] has built a consistent estimator for this model using the kernel method. They established asymptotic properties, especially its quadratic convergence, in the case where the observations are independent and identically distributed.

The literature on the relative error regression (RER) in nonparametric functional data analysis is still not very developed. The first consistent results were obtained by [2], where relative regression was used as a classification tool. For the kernel method combined with the local linear method, [6] gives the asymptotic properties of the nonparametric prediction via relative error regression. Recently, [1] proposed a kernel regression estimator version in the spatial framework context and derived asymptotically and numerically the effectiveness of this kind of estimator, whereas [4] proposed a functional version of the relative kernel regression estimator while [8] proposed a nonparametric method estimation for deconvolution regression model using relative error prediction.

2. Model

In the censoring case, instead of observing the lifetimes Y (which has a continuous distribution function (df) H) we observe the censored lifetimes of items. That is, assuming that $(C_i)_{1\leq i\leq n}$ is a sequence of i.i.d. censoring random variable (r.v.) with common unknown continuous df G. Then, in the right censorship model, we only observe the n pairs (T_i, δ_i) with

$$T_i = Y_i \wedge C_i \ and \ \delta_i = \mathbb{1}_{\{Y_i\leq C_i\}}, 1 \leq i \leq n, \tag{4}$$

where $\mathbb{1}_A$ denotes the indicator function of the set A.

Now, we assume that $(C_i)_{1\leq i\leq n}$ and $(X_i, Y_i)_{1\leq i\leq n}$ are independent. In censorship model, only the $(X_i, T_i, \delta_i)_{1\leq i\leq n}$ are observed. For any df L, we will write $\tau_L = sup\{t : \bar{L}(t) > 0\}$, where $\bar{L}(.) = 1 - L(.)$ On the other hand, $L_n(.)$ will denote a functional estimator of $L(.)$. Denote by $\widetilde{r}(x)$ the estimator of $r(x)$ in presence of censored data. Then,

$$\widetilde{r}(x) = \frac{\sum_{i=1}^{n} \frac{\delta_i T_i^{-1}}{\bar{G}(T_i)} K\left(\frac{d(x,X_i)}{h}\right)}{\sum_{i=1}^{n} \frac{\delta_i T_i^{-2}}{\bar{G}(T_i)} K\left(\frac{d(x,X_i)}{h}\right)} =: \frac{\widetilde{g}_1(x)}{\widetilde{g}_2(x)}, \tag{5}$$

where

$$\widetilde{g}_l(x) = \frac{\sum_{i=1}^{n} \frac{\delta_i T_i^{-l}}{\bar{G}(T_i)} K\left(\frac{d(x,X_i)}{h}\right)}{n\mathbb{E}\left[K\left(\frac{d(x,X_1)}{h}\right)\right]}, \qquad \text{for} \quad l = 1,2$$

In practice, G is unknown. So, we use the Kaplan-Meier estimator in [7] of $\bar{G}$ given by

$$\bar{G}_n(t) = \begin{cases} \prod_{i=1}^{n} \left(1 - \frac{1-\delta_{(i)}}{n-i+1}\right)^{\mathbb{1}_{\{T_{(i)} \le t\}}} & \text{if } t \le T_{(n)} \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where $T_{(1)} \le T_{(2)} \le ... \le T_{(n)}$ are the order statistics of $(T_i)_{1 \le i \le n}$ and $\delta_{(i)}$ is the concomitant of $T_{(i)}$. Therefore, the estimator of $r(x)$ is given by

$$\widetilde{r}_n(x) = \frac{\sum_{i=1}^{n} \frac{\delta_i T_i^{-1}}{\bar{G}_n(T_i)} K\left(\frac{d(x,X_i)}{h}\right)}{\sum_{i=1}^{n} \frac{\delta_i T_i^{-2}}{\bar{G}_n(T_i)} K\left(\frac{d(x,X_i)}{h}\right)} =: \frac{\widetilde{g}_{1,n}(x)}{\widetilde{g}_{2,n}(x)}, \tag{7}$$

where

$$\widetilde{g}_{l,n}(x) = \frac{1}{n\mathbb{E}\left[K\left(\frac{d(x,X_1)}{h}\right)\right]} \sum_{i=1}^{n} \frac{\delta_i T_i^{-l}}{\bar{G}_n(T_i)} K\left(\frac{d(x,X_i)}{h}\right) \qquad \text{for} \quad l = 1,2.$$

3. **Assumptions and main results**

The main purpose of this section is to study the uniform almost-complete convergence[1](a.co.) of $\widetilde{r}_n(x)$ toward $r(x)$.

From now on, for all x in $\mathcal{F}$, for all positive real h, and denote by $\mathcal{N}_x$ the neighborhood of the point x, when no confusion is possible, we will denote by c and c' generic constants and define $K_i(x)$ by

$$K_i(x) = K\left(\frac{d(x,X_i)}{h}\right) \; for \; i = 1, ..., n,$$

where K is a kernel function and $h := h_{n,K}$ is a sequence of positive numbers decreasing toward 0. We will also use the notation

$$\varphi_x(h) = \mathbb{P}(X \in B(x,h)), \tag{8}$$

where $B(x,h) = \{x' \in \mathcal{F}, d(x,x) \le h\}$.

[1]Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of real r.v.'s. We say that Z_n converges almost completely (a.co.) toward zero if and only if $\forall \varepsilon > 0, \sum_{n=1}^{\infty} \mathbb{P}(|Z_n| > \varepsilon) < \infty$. Moreover, we say that the rate of the almost complete convergence of Z_n to zero is of order u_n (with $u_n \to 0$) and we write $Z_n = O(u_n)$ *a.co.* if and only if $\exists \varepsilon > 0$ such that $\sum_{n=1}^{\infty} \mathbb{P}(|Z_n| > \varepsilon u_n) < \infty$. This kind of convergence implies both almost sure convergence and convergence in probability.

We recall the definition of the Kolmogorov's entropy which is an important tool to obtain uniform convergence results. Given a subset $S_{\mathcal{F}} \subset S$ and $\varepsilon > 0$, denote $N_\varepsilon(S)$ or N the minimal number of open balls of radius ε needed to cover S. Then, the quantity $\psi_{S_{\mathcal{F}}} = \log(N)$ is called Kolmogorov's ε-entropy of the set S. In what follows, we will need the following assumptions:

(H1): $\mathbb{P}(X \in B(x,h)) =: \varphi_x(h) > 0$ for all $h > 0$ and $\lim_{h\to 0} \varphi_x(h) = 0$.

(H2): For all $(x_1, x_2) \in \mathcal{N}_x^2$, we have

$$|g_l(x_1) - g_l(x_2)| \leq c d^{k_l}(x_1, x_2) \; for \; k_l > 0.$$

(H3): The kernel K is a bounded Lipshitzian and differentiable function on its support $(0;1)$ and satisfying:

$$0 < c \leq K(.) \leq c' < +\infty,$$

and its first derivative function K' is such that: $-\infty < c < K'(.) < c' < 0$.

(H4): The bandwidth h satisfies:

(i): $\sqrt{\frac{\log\log n}{n}} = o(\varphi_x(h))$;

(ii): $\frac{n\varphi_x(h)}{\log n} \to \infty \; as \; n \to \infty$.

(H5): The response variable Y is such that: $|Y| > c > 0$ for all $x \in \mathcal{F}$ and

$$\inf_{x\in\mathcal{F}} g_2(x) \geq \gamma > 0.$$

(H6): The functions φ_x and $\psi_{S_{\mathcal{F}}}$ are such that:

(H6a): there exists $\eta_0 > 0$ such that for all $\eta < \eta_0$, $\varphi'_x(\eta) < c$, where φ'_x denotes the first derivative function of φ_x.

(H6b): for a large enough integer n, we have:

$$\frac{(\log n)^2}{n\varphi_x(h)} < \psi_{S_{\mathcal{F}}}\left(\frac{\log n}{n}\right) < \frac{n\varphi_x(h)}{\log n},$$

(H6c): the Kolmogorov's ε-entropy of $S_{\mathcal{F}}$ satisfies:

$$\sum_{n=1}^{\infty} \exp\left[(1-\beta)\psi_{S_{\mathcal{F}}}\left(\frac{\log n}{n}\right)\right] < \infty \text{ for some } \beta > 1.$$

Now we are in a position to give our main result.

Theorem 3.1. *Under Assumptions **(H1)-(H6)**, we have*

(9)

$$\sup_{x\in\mathcal{F}} |\widetilde{r}_n(x) - r(x)| = O_{a.co.}\left(h^{k_1}\right) + O_{a.co.}\left(h^{k_2}\right) + O_{a.co.}\left(\sqrt{\frac{\psi_{S_{\mathcal{F}}}\left(\frac{\log n}{n}\right)}{n\varphi_x(h)}}\right).$$

3.1. **Proofs of Theorem 3.1.** From (9), we can see that:

$$\begin{aligned}
\sup_{x\in\mathcal{F}} |\widetilde{r}_n(x) - r(x)| \leq & \sup_{x\in\mathcal{F}} \left\{ \left| \frac{\widetilde{g}_{1,n}(x)}{\widetilde{g}_{2,n}(x)} - \frac{\widetilde{g}_1(x)}{\widetilde{g}_{2,n}(x)} \right| + \left| \frac{\widetilde{g}_1(x)}{\widetilde{g}_{2,n}(x)} - \frac{\mathbb{E}(\widetilde{g}_1(x))}{\widetilde{g}_{2,n}(x)} \right| \right. \\
+ & \left. \left| \frac{\mathbb{E}(\widetilde{g}_1(x))}{\widetilde{g}_{2,n}(x)} - \frac{g_1(x)}{\widetilde{g}_{2,n}(x)} \right| + \left| \frac{g_1(x)}{\widetilde{g}_{2,n}(x)} - \frac{g_1(x)}{g_2(x)} \right| \right\} \\
\leq & \frac{1}{\inf_{x\in\mathcal{F}} |\widetilde{g}_{2,n}(x)|} \left\{ \sup_{x\in\mathcal{F}} |\widetilde{g}_{1,n}(x) - \widetilde{g}_1(x)| + \sup_{x\in\mathcal{F}} |\widetilde{g}_1(x) - \mathbb{E}(\widetilde{g}_1(x))| \right. \\
+ & \left. \sup_{x\in\mathcal{F}} |\mathbb{E}(\widetilde{g}_1(x)) - g_1(x)| \right\} + \frac{\sup_{x\in\mathcal{F}} |g_1(x)|\gamma^{-1}}{\inf_{x\in\mathcal{F}} |\widetilde{g}_{2,n}(x)|} \left\{ \sup_{x\in\mathcal{F}} |\widetilde{g}_{2,n}(x) - \widetilde{g}_2(x)| \right. \\
+ & \left. \sup_{x\in\mathcal{F}} |\widetilde{g}_2(x) - \mathbb{E}(\widetilde{g}_2(x))| + \sup_{x\in\mathcal{F}} |\mathbb{E}(\widetilde{g}_2(x)) - g_2(x)| \right\}.
\end{aligned}$$

Therefore, Theorem 3.1's result is a consequence of the following intermediate results, where their proofs are postponed to the appendix.

Lemma 3.2. *Under assumptions **(H2)-(H5)**, we have*

$$\sup_{x\in\mathcal{F}} |\widetilde{g}_{l,n}(x) - \widetilde{g}_l(x)| = O_{a.s.}\left(\sqrt{\frac{\log\log n}{n}}\right), \text{ with } l \in \{1,2\}. \tag{10}$$

Where $O_{a.s.}$ means the rate of the almost sure convergence.

Lemma 3.3. *Under assumptions **(H1)-(H3)** and **(H5)**, we have*

$$\sup_{x\in\mathcal{F}} |\mathbb{E}(\widetilde{g}_l(x)) - g_l(x)| = O\left(h^{k_l}\right), \tag{11}$$

withe $l \in \{1,2\}$.

Lemma 3.4. *Under assumptions **(H1)-(H3)** and **(H6)**, we have*

$$\sup_{x\in\mathcal{F}} |\widetilde{g}_l(x) - \mathbb{E}(\widetilde{g}_l(x))| = O_{a.co.}\left(\sqrt{\frac{\psi_{S_\mathcal{F}}\left(\frac{\log n}{n}\right)}{n\varphi_x(h)}}\right), \tag{12}$$

withe $l \in \{1,2\}$.

Corollary 3.5. *Under the assumptions of lemma 3.3 and 3.4, we obtain:*

$$\text{there exists } \delta > 0 \text{ such that } \sum_{n=1}^{\infty} \mathbb{P}\left(\inf_{x\in\mathcal{F}} |\widetilde{g}_{2,n}(x)| < \delta\right) < \infty. \tag{13}$$

4. Simulation study

In order to see the behavior of our proposed estimator, we consider the curves generated in the following way:

$$X_i(t) = a_i \sin(4(b_i - t)) + b_i + \eta_{i,t} \;\; i = 1:200 \;\; t \in [0,1[,$$

where $a_i \sim \mathcal{N}(5,2)$, $b_i \sim \mathcal{N}(0,\ 0.1)$ and $\eta_{i,t} \sim \mathcal{N}(0,\ 0.2)$. All the curves are discretized on the same grid generated from $m = 150$ equispaced points $t \in [0,1[$. The observations Y_i's for $i = 1, ..., n$ are generated from the model

$$Y_i = r(X_i) + \epsilon_i \text{ where } \epsilon_i \sim \mathcal{N}(0,\ 0.01),$$

where

$$r(x) = \int_0^1 \frac{dt}{1 + |x(t)|}.$$

In practice, the semi-metric choice is based on the regularity of the curves which are under study. In our case, regarding the shape of the curves X_i, it is clear that the PCA-type semi-metric (cf. [5]) is well adapted to this data set. It should also be noticed that the best results concerning prediction are obtained for $q = 4$ (the number of components in the PCA-type semi-metric). The optimal bandwidth h is chosen by the cross-validation method for the k nearest neighbors (kNN) in a local way.
We select the quadratic kernel for both classic and relative estimators defined by

$$K(u) = \frac{3}{2}(1 - u^2)\mathbb{1}_{(0,1)}.$$

Next, we consider a sample of size $n = 200$ and we split the data generated from the model above into two subsets: a training sample $(X_i, T_i, \delta_i), i = 1, ..., 150$ and a test sample $(X_i, T_i, \delta_i), i = 151, ..., 200$. Then, we calculate the estimator $\hat{\theta}(X_i)$ for any $i \in \{151, ..., 200\}$.

We also, simulate n i.i.d. rv's C_i, $i = 1, ..., n$ with law $\mathcal{E}(\lambda)$ (the exponential law with the λ parameter that controls the censorship rate).

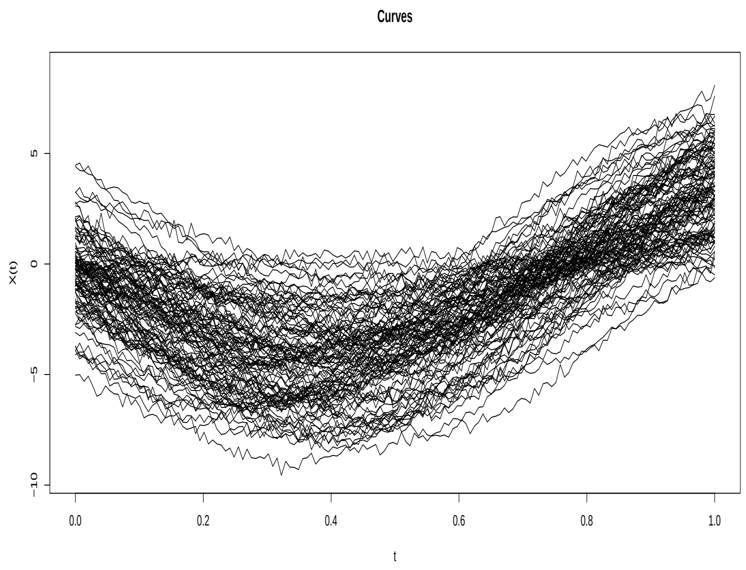

FIGURE 1. The curves $X_{i=1,...,100}(t),\ t \in [0, 1[$.

The performance of both estimators was compared under the mean squared prediction error (MSE) criterion:

$$MSE = \frac{1}{50} \sum_{j=151}^{200} (\theta(X_j) - \hat{\theta}(X_j))^2,$$

where $\hat{\theta}(X_j)$ means the estimator of both regression models and $\theta(X_j)$ the response variable.

1) Data without outliers : The obtained results are shown in Figure 2. With the censorship rate $CR = 1.33\%$, it is clear that there is no meaningful difference between the two estimation methods: the Classical Kernel Estimator (CKE) and the Relative Error Estimator (REE) ($MSE_{CKE} = 0.00038$, $MSE_{REE} = 0.00048$).

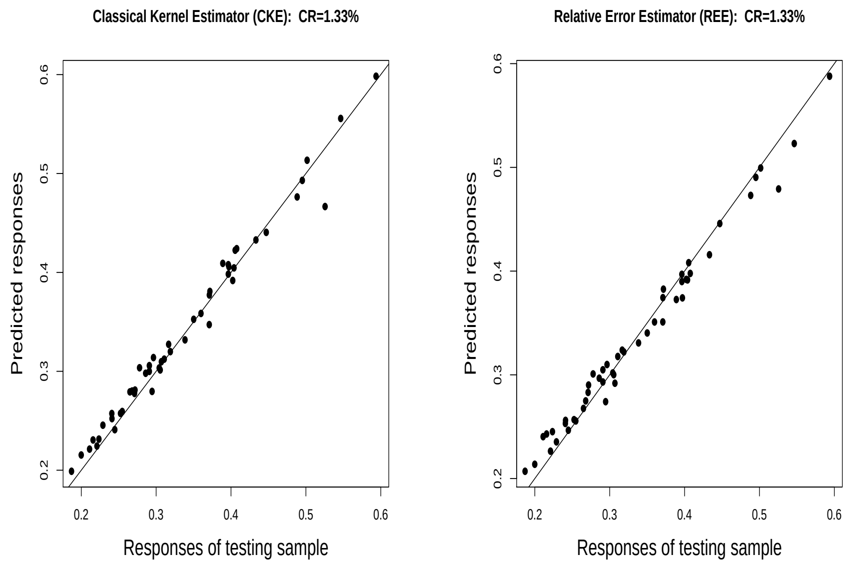

FIGURE 2. comparison between the Classical Kernel Estimator (CKE) and the Relative Error Estimator (REE) without outliers.

2) Data with outliers : Here, we concentrate on the comparison of both models' performances in the presence of outliers. For this aim, we introduce artificial outliers by multiplying some values of Y in the training sample by 10. The estimators of both models are obtained by the same previous selection methods of the smoothing parameter, i.e., the same metric d and also the same kernel K. Finally, the obtained results are shown in Table 1 and displayed in Figure 3. Note that, in Figure 2 the two estimators are equivalent but in Figure 3, in which we considered the presence

TABLE 1. MSE for the Classical Kernel Estimator (CKE) and the Relative Error Estimator (REE) according to numbers of introduced artificial outliers.

Number of artificial outliers	0	10	30	50
Classical Kernel Estimator MSE_{CKE}	0.00076	0.02520	3.98068	434.82333
Relative Error Estimator MSE_{REE}	0.00060	0.00064	0.00072	0.00054

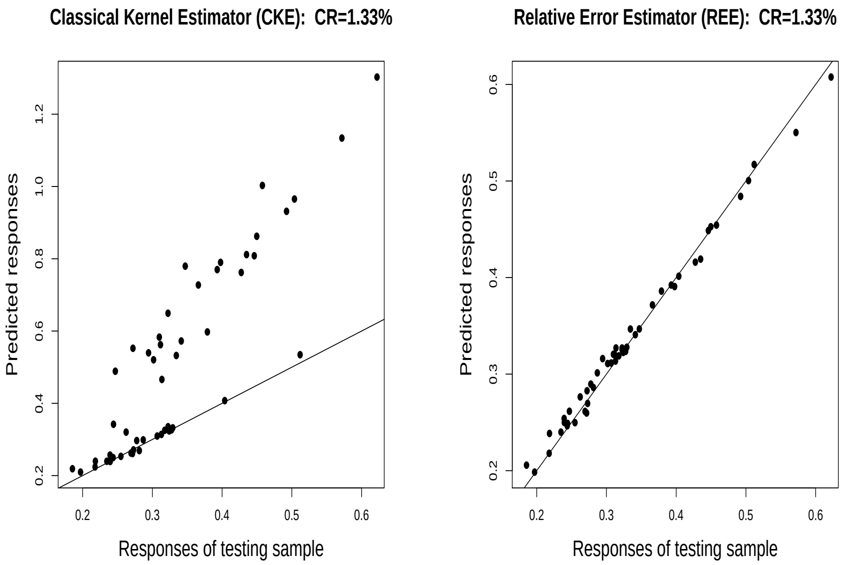

FIGURE 3. comparison between the Classical Kernel Estimator (CKE) and the Relative Error Estimator (REE) in the presence of outliers.

of outliers, the relative error regression is robust than the classical kernel regression; i.e., the classical kernel method is susceptible to the presence of outliers. Now, we will study the behavior of our estimator with different censored rates (CR). The results are shown in Table 2. We see that the quality of fit is affected and becomes worse as the CR increases, but the relative error estimator is more efficient than the classical one in the presence of censoring data.

TABLE 2. MSE for the Classical Kernel Estimator (CKE) and the Relative Error Estimator (REE) according to to the censoring rates with different sample size.

Sample size	CR	MSE_{CKE}	MSE_{REE}
100	10%	0.00239	0.00203
	20%	0.00613	0.00366
	60%	0.01483	0.00679
200	10%	0.00175	0.00182
	20%	0.00545	0.00292
	60%	0.01155	0.00576
600	10%	0.00100	0.00051
	20%	0.00860	0.00284
	60%	0.01179	0.00408

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